

Perturbations and Stability of Static Black Holes in Higher Dimensions

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In this chapter we consider perturbations and stability of higher dimensional black holes focusing on the static background case. We first review a gauge-invariant formalism for linear perturbations in a fairly generic class of $(m+n)$ -dimensional spacetimes with a warped product metric, including black hole geometry. We classify perturbations of such a background into three types, the tensor, vector and scalar-type, according to their tensorial behavior on the n -dimensional part of the background spacetime, and for each type of perturbations, we introduce a set of manifestly gauge invariant variables. We then introduce harmonic tensors and write down the equations of motion for the expansion coefficients of the gauge invariant perturbation variables in terms of the harmonics. In particular, for the tensor-type perturbations a single master equation is obtained in the $(m+n)$ -dimensional background, which is applicable for perturbation analysis of not only static black holes but also some class of rotating black holes as well as black-branes. For the vector and scalar type, we derive a set of decoupled master equations when the background is a $(2+n)$ -dimensional static black hole in the Einstein-Maxwell theory with a cosmological constant. As an application of the master equations, we review the stability analysis of higher dimensional charged static black holes with a cosmological constant. We also briefly review the recent results of a generalization of the perturbation formulae presented here and stability analysis to static black holes in generic Lovelock theory.

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§1. Introduction

There is a large variety of black hole solutions in higher dimensions. The stability of such exact solutions is clearly an important issue, and analysing linear perturbations of the existing exact solutions would be the first step to take. If a stationary black hole solution is shown to be stable under perturbations, it implies that the solution describes a possible final state of dynamical evolution of a gravitating system. If, on the other hand, an instability is found, it then indicates the existence of a different branch of solutions, which the original solution may decay into, and one can then anticipate more variety of black hole solutions.

Apart from the stability issue, perturbation analysis also tells us a lot about basic properties of black hole solutions. The spectra of quasinormal modes¹⁾ contain information about the geometric structure of the background metric, especially near the horizon. The study of stationary perturbations of a stationary black hole solution provides a criterion for the uniqueness/non-uniqueness property of the solution. Considering such stationary perturbations of a known solution may also be useful when attempting to construct approximate solutions.

In this article, we shall review the perturbation formulae for higher dimensional static black holes and the stability analysis, following, to a large extent, Refs. 2)–4). The linearised Einstein equations off of a black hole spacetime are in general still quite involved, having a number of perturbation variables intricately coupled, and one therefore needs to simplify them to a tractable form. In 4-dimensions, the perturbed Einstein equations for a stationary vacuum black hole solution can be reduced to a set of simple decoupled ordinary differential equations (ODEs) by exploiting some particular geometric feature of the background solution: the Regge-Wheeler-Zerilli equations^{5),6)} for the Schwarzschild metric and Teukolsky equations⁷⁾ for the Kerr metric. For higher dimensional rotating black holes, we are still a long way from having such a complete formulation for perturbations, though considerable progress along this direction has recently been made for some special cases^{8)–11)} [see Chapter 7 and references therein for the rotating black hole case]. Fortunately,

for static black holes, e.g., Schwarzschild-Tangherlini metric and its cousins, such a reduction is possible in arbitrary higher dimensions, and a set of decoupled master equations, which correspond to the Regge-Wheeler-Zerilli equations in 4-dimensions, are now available.^{2),3)} More precisely, consider a $(2 + n)$ -dimensional static black hole in Einstein-Maxwell system with a cosmological constant as our background, in which the n -dimensional internal space is an Einstein space and describes the horizon cross-section geometry. Then perturbation variables are classified into three types according to their tensorial behavior on the internal space. For each type of perturbations, a basis of gauge-invariant variables is introduced, and the Einstein and Maxwell equations are written in terms of them. For each type of perturbations, the perturbed equations of motion are further reduced to a set of decoupled master equations for a single scalar variable on the 2-dimensional part of the background spacetime. Furthermore, the master equations thus obtained in Refs. 2), 3) take the form of a second-order self-adjoint ODE with respect to the radial coordinate of the black hole background and are therefore immediately applied to the stability analysis.^{3),4)}

In the next section we first establish our notation and convention by describing our background geometry. We next explain how to decompose tensor fields in our background spacetime and how to construct manifestly gauge-invariant perturbation variables. In section 3, we introduce harmonic tensors on n -dimensional Einstein space and provide several theorems concerning basic properties of the harmonic tensors. We also give explicit expressions for scalar, vector and tensor harmonics on S^n in terms of the homogeneous coordinates for S^n in E^{n+1} . Subsequently from section 4 to section 6, we focus on perturbations of static black holes in the Einstein-Maxwell system with cosmological constant and derive a set of master equations. In section 7, we describe how to derive the master equations for static black holes in most general Lovelock theory recently obtained.¹²⁾ Then in section 8, using the master equations, we show that a large class of static black holes in the Einstein-Maxwell system with cosmological constant are stable with respect to linear gravitational as well as electromagnetic perturbations. We also briefly comment on the recent study of instability of Lovelock black holes. Section 9 is devoted to summary and discussion.

§2. Notation and conventions

In this section we describe our background spacetime and discuss how to classify tensor fields on the spacetime. We then discuss the problem of gauge freedom and introduce gauge-invariant variables. We generally follow the notation and conventions of the papers 2), 4), and 3), which throughout this chapter we refer to as Papers I, II, and III, respectively.

2.1. Background Geometry

We consider an $(m + n)$ -dimensional spacetime whose manifold structure is locally a warped product type, $\mathcal{M} = \mathcal{N}^m \times \mathcal{K}^n$, and accordingly we often need to distinguish between tensors living in these different manifolds, \mathcal{M} , \mathcal{N}^m , and \mathcal{K}^n .

For this reason, we do not employ the abstract index notation¹³⁾ in this chapter, and instead, we use upper case latin indices in the range K, L, M, N, \dots to denote tensors on \mathcal{M} , lower case latin indices in the range a, b, \dots, h on \mathcal{N}^m , and lower case latin indices in the range i, j, \dots, p on \mathcal{K}^n . Accordingly, we introduce coordinates $x^M = (y^a, z^i)$ in terms of which our background metric is written

$$g_{MN}dx^Mdx^N = g_{ab}(y)dy^ady^b + r^2(y)\gamma(z)_{ij}dz^idz^j. \quad (2.1)$$

We assume that an m -dimensional metric $g_{ab}(y)$ on \mathcal{N}^m is Lorentzian and an n -dimensional internal metric $\gamma(z)_{ij}$ on \mathcal{K}^n is *Einstein* i.e.,

$$\hat{R}_{ij} = (n-1)K\gamma_{ij} \quad (2.2)$$

for some constant K , with \hat{R}_{ij} being the Ricci tensor of γ_{ij} . When \mathcal{K}^n is maximally symmetric, the constant K corresponds to the sectional curvature of \mathcal{K}^n , and in what follows we normalize $K = 0, \pm 1$. We assume that \mathcal{K}^n be complete, as it describes the geometry of cross-sections of the event horizon.

We denote by ∇_M , D_a , and \hat{D}_i , the covariant derivative operators compatible with g_{MN} , g_{ab} , and γ_{ij} , respectively. Having these derivative operators, we can define the curvature tensors on \mathcal{M} , \mathcal{N}^m and \mathcal{K}^n , and find their relations in terms of the coordinate components as

$$R^a{}_{bcd} = {}^mR^a{}_{bcd}, \quad (2.3a)$$

$$R^a{}_{ibj} = -\frac{D^a D_b r}{r} g_{ij}, \quad (2.3b)$$

$$R^i{}_{jkl} = \hat{R}^i{}_{jkl} - (Dr)^2(\delta_k^i \gamma_{jl} - \delta_l^i \gamma_{jk}), \quad (2.3c)$$

where ${}^mR^a{}_{bcd}$ and $\hat{R}^i{}_{jkl}$ are the curvature tensors of g_{ab} and γ_{ij} , respectively. From this and Eq. (2.2), we obtain

$$R_{ab} = \frac{1}{2} {}^mR g_{ab} - n \frac{D_a D_b r}{r}, \quad (2.4a)$$

$$R_{ai} = 0, \quad (2.4b)$$

$$R_{ij} = \left(-\frac{\square r}{r} + (n-1) \frac{K - (Dr)^2}{r^2} \right) g_{ij}, \quad (2.4c)$$

$$R = {}^mR - 2n \frac{\square r}{r} + n(n-1) \frac{K - (Dr)^2}{r^2}, \quad (2.4d)$$

where $\square = D^a D_a$ and mR is the scalar curvature of g_{ab} . Note that the Ricci tensor takes the same form as in the case in which \mathcal{K}^n is maximally symmetric.¹⁴⁾

The geometric structure of our background spacetime requires that the background stress-energy tensor T_{MN} should take the form

$$T_{ai} = 0, \quad T^i{}_j = P \delta^i{}_j, \quad (2.5)$$

where P is a scalar field on \mathcal{N}^m .

Then, the background Einstein equations, including cosmological constant, Λ , are written as

$$mG_{ab} - n \frac{D_a D_b r}{r} - \left(\frac{n(n-1)}{2} \frac{K - (Dr)^2}{r^2} - n \frac{\square r}{r} \right) g_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab}, \quad (2.6)$$

$$-\frac{1}{2} m R - \frac{(n-1)(n-2)}{2} \frac{K - (Dr)^2}{r^2} + (n-1) \frac{\square r}{r} = \kappa^2 P - \Lambda, \quad (2.7)$$

where κ^2 denotes the gravitational constant. Note that although our main concern is about static black holes for which $m = 2$ with the metric form given by Eq. (2.14) below, our metric ansatz above also allows us to consider various different geometries, such as black-string/branes when $m \geq 3$ and Myers-Perry black holes with a single rotation when $m = 4$.

2.2. Static black holes in $(2+n)$ -dimensions

Now, having static black hole geometry in mind, let us set $m = 2$ and consider an electromagnetic field \mathcal{F}_{MN} as a source for the background gravitational field. The field strength \mathcal{F}_{MN} may be given by

$$\mathcal{F} = \frac{1}{2} E_0 \epsilon_{ab} dy^a \wedge dy^b + \frac{1}{2} \mathcal{F}_{ij} dz^i \wedge dz^j. \quad (2.8)$$

Then, from $\nabla_{[M} \mathcal{F}_{NL]} = 0$, we obtain

$$E_0 = E_0(y), \quad \mathcal{F}_{ij} = \mathcal{F}_{ij}(z), \quad \partial_{[k} \mathcal{F}_{ij]} = 0, \quad (2.9)$$

and from $\nabla_N \mathcal{F}^{MN} = 0$,

$$0 = \nabla_N \mathcal{F}^{aN} = \frac{1}{r^n} \epsilon^{ab} D_b (r^n E_0), \quad (2.10a)$$

$$0 = \nabla_N \mathcal{F}^{iN} = \hat{D}_j \mathcal{F}^{ij}. \quad (2.10b)$$

These equations imply that the electric field E_0 takes the Coulomb form,

$$E_0 = \frac{q}{r^n}, \quad (2.11)$$

and $\hat{\mathcal{F}} = \frac{1}{2} \mathcal{F}_{ij}(z) dz^i \wedge dz^j$ is a harmonic form on \mathcal{K}^n . Although, in general, there may exist such a harmonic form that produces an energy-momentum tensor consistent with the structure of the Ricci tensors in Eq. (2.4), in the following we consider only the case $\mathcal{F}_{ij} = 0$.

With this assumption, the energy-momentum tensor for the electromagnetic field,

$$T_{MN}^{(\text{em})} = \mathcal{F}_{ML} \mathcal{F}_N{}^L - \frac{1}{4} g_{MN} \mathcal{F}_{LK} \mathcal{F}^{LK}, \quad (2.12)$$

is written

$$T^{(\text{em})a}{}_b = -P \delta^a{}_b, \quad T^{(\text{em})i}{}_j = P \delta^i{}_j; \quad P = \frac{1}{2} E_0^2 = \frac{q^2}{2r^{2n}}. \quad (2.13)$$

Then, solving the background Einstein equations, we have, when $\nabla r \neq 0$, the black hole type solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma_n^2, \quad (2.14)$$

where

$$f(r) = K - \lambda r^2 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}}, \quad (2.15)$$

and

$$\lambda := \frac{2A}{n(n+1)}, \quad Q^2 := \frac{\kappa^2 q^2}{n(n-1)}. \quad (2.16)$$

The spacetime described by this metric can contain a regular black hole for some restricted ranges of the parameters, M , Q , λ , K . [For the allowed parameter regions for regular black holes, see Appendix A of Paper III.]

Note that we can also consider, as our background, solutions with $\nabla r = 0$, for which \mathcal{M} becomes a cartesian product of a 2-dimensional maximally symmetric spacetime \mathcal{N}^2 and the Einstein space \mathcal{K}^n . Such solutions include the Nariai solution,^{15), 16)} as well as the Bertotti-Robinson solution in 4-dimensions. The solutions, and their parameter ranges are given in Paper III and also in Ref. 17). As in Paper III, we can study perturbations of such Nariai-type solutions in a similar manner as in the black hole background case, but in this chapter, we are not going to deal with this case.

2.3. Decomposition of vectors and symmetric tensors on compact manifolds

When considering perturbations, we will in general have to deal with two major issues. One is concerning the ambiguity in a choice of gauge, due to the invariance of the Einstein equations under an infinitesimal gauge transformation. This issue will be discussed in the next subsection. The other one is that even when linearised, the Einstein equations are still a set of intricately coupled equations for a number of perturbation variables of the metric and matter fields, and are in general very difficult to solve. We therefore need to reduce the linearised Einstein equations to a simple tractable form. For this purpose we first classify perturbation variables into three different types according to their tensorial behavior on \mathcal{K}^n in such a way that the linearised Einstein equations get decoupled and can be dealt with separately for each type of perturbations. We then introduce harmonic tensors on \mathcal{K}^n so that each type of the perturbed Einstein equations reduces to a set of equations on the m -dimensional spacetime (\mathcal{N}^m, g_{ab}) . This procedure enables us to obtain a set of significantly simplified equations for master scalar variables as we will see in later sections.

Let us consider how to classify perturbation variables. We first note the following two decomposition theorems:

(i) Suppose $(\mathcal{K}^n, \gamma_{ij})$ be a compact Riemannian manifold. Any dual vector field on \mathcal{K}^n can be uniquely decomposed as

$$v_i = V_i + \hat{D}_i S \quad (2.17)$$

where $\hat{D}^i V_i = 0$. This is essentially the well-known Hodge decomposition theorem, and we refer to V_i and S , respectively, as the vector- and scalar-type components of the dual vector v_i .

(ii) Suppose $(\mathcal{K}^n, \gamma_{ij})$ be a compact Riemannian Einstein space, $\hat{R}_{ij} = c\gamma_{ij}$ for some constant c . Any second rank symmetric tensor field t_{ij} can be uniquely decomposed as

$$t_{ij} = t_{ij}^{(2)} + 2\hat{D}_{(i} t_{j)}^{(1)} + t_L \gamma_{ij} + \hat{L}_{ij} t_T, \quad (2.18)$$

$$\hat{L}_{ij} := \hat{D}_i \hat{D}_j - \frac{1}{n} \gamma_{ij} \hat{\Delta} \quad (2.19)$$

where $\hat{D}^i t_{ij}^{(2)} = 0$, $t^{(2)i}_i = 0$, $\hat{D}^i t^{(1)}_i = 0$, and $t_L = t^m_m/n$. We refer to $t_{ij}^{(2)}$, $t_i^{(1)}$, and (t_T, t_L) , respectively, as the tensor-, vector- and scalar-type components of t_{ij} . Note that tensor component $t_{ij}^{(2)}$ exists only when $n \geq 3$.

A similar decomposition theorem—in which \mathcal{K}^n is considered to be maximally symmetric—has been proved by Kodama and Sasaki.¹⁸⁾ A general proof of the above theorems (i) and (ii) is found in Ref. 19), in which the compactness of \mathcal{K}^n is used in an essential way to show that any symmetric elliptic operator, such as $\hat{D}^i \hat{D}_{(i} V_{j)}$ on V_i , is essentially self-adjoint and the spectrum of any self-adjoint operator is discrete. The symmetric, elliptic operators appeared in the proof will be essentially self-adjoint on many non-compact manifolds of interests, even though their spectra will be continuous, and analogous decomposition theorems should also hold for many non-compact manifolds. In the following when considering the case in which \mathcal{K}^n is non-compact, we simply assume that analogous decomposition results hold.

Now let us consider metric perturbations $h_{MN} = \delta g_{MN}$ on our background spacetime (\mathcal{M}, g_{MN}) . We may project h_{MN} relative to the Einstein manifold \mathcal{K}^n as

$$h_{MN} dx^M dx^N = h_{ab} dy^a dy^b + 2h_{ai} dy^a dz^i + h_{ij} dz^i dz^j. \quad (2.20)$$

The component h_{ab} is purely scalar with respect to transformations on \mathcal{K}^n . As for the components h_{ai} and h_{ij} , applying the above decomposition theorems (i) (ii), we further decompose them into their scalar, vector and tensor parts with respect to \mathcal{K}^n as

$$h_{ai} = \hat{D}_i h_a + h_{ai}^{(1)}, \quad (2.21)$$

$$h_{ij} = h_T^{(2)}{}_{ij} + 2\hat{D}_{(i} h_T^{(1)}{}_{j)} + h_L \gamma_{ij} + \hat{L}_{ij} h_T^{(0)}, \quad (2.22)$$

where

$$\hat{D}^j h_T^{(2)}{}_{ij} = h_T^{(2)i}{}_i = 0, \quad (2.23)$$

$$\hat{D}^a h_{ai}^{(1)} = 0, \quad \hat{D}^i h_T^{(1)}{}_i = 0. \quad (2.24)$$

Thus, the tensor part of h_{MN} is $h_T^{(2)}{}_{ij}$, the vector part of h_{MN} consists of $(h_{ai}^{(1)}, h_T^{(1)}{}_i)$, and the scalar part of h_{MN} consists of $(h_{ab}, h_a, h_L, h_T^{(0)})$.

Similarly, we can decompose perturbations of the energy-momentum tensor, δT_{MN} , into the tensor part $\delta T_{Tij}^{(2)}$, the vector part $(\delta T_{ai}^{(1)}, \delta T_{Ti}^{(1)})$, and the scalar part $(\delta T_{ab}, \delta T_a, \delta T_L, \delta T_T)$, where

$$\delta T_{ai} = \hat{D}_i \delta T_a + \delta T_{ai}^{(1)}, \quad (2.25)$$

$$\delta T_{ij} = \delta T_{Tij}^{(2)} + 2\hat{D}_{(i} \delta T_T^{(1)}{}_{j)} + \delta T_L \gamma_{ij} + \hat{L}_{ij} \delta T_T, \quad (2.26)$$

with $\hat{D}^i \delta T_T^{(2)}{}_{ij} = 0$, $\delta T_T^{(2)}{}_{i i} = 0$, $\hat{D}^i \delta T_{ai}^{(1)} = 0 = \hat{D}^i \delta T_T^{(1)}{}_{i i}$.

The linearised Einstein equations are decomposed into these three types, and thus we can deal with perturbations of each type separately. Before writing down the equations for each type of perturbations, we discuss the gauge issue.

2.4. Gauge-invariant formulation

As is well-known, the Einstein equations are invariant under gauge transformation generated by an (infinitesimal) vector field ξ^M . Accordingly, a perturbation variable and its gauge transformed one, such as δg_{MN} and $\delta g_{MN} - \mathcal{L}_\xi g_{MN}$, both must describe the same physical situation, hence giving rise to an ambiguity in the representation of perturbation variables. In order to remove this gauge ambiguity and extract the physical degrees of freedom, we may proceed either by imposing appropriate conditions that completely fix the gauge freedom, or by constructing manifestly gauge-invariant variables and writing down the relevant equations in terms of them. The two approaches are equivalent. To see this, let us have a look at gauge transformation law of perturbation variables.

In terms of the coordinates (y^a, z^i) , the gauge transformation of the metric perturbation $h_{MN} = \delta g_{MN}$ is written as

$$h_{ab} \rightarrow h_{ab} - D_a \xi_b - D_b \xi_a, \quad (2.27)$$

$$h_{ai} \rightarrow h_{ai} - r^2 D_a \left(\frac{\xi_i}{r^2} \right) - \hat{D}_i \xi_a, \quad (2.28)$$

$$h_{ij} \rightarrow h_{ij} - 2\hat{D}_{(i} \xi_{j)} - 2r(D^a r) \xi_a \gamma_{ij}. \quad (2.29)$$

Similarly, the gauge transformation of perturbation of the energy-momentum tensor is given by

$$\delta T_{ab} \rightarrow \delta T_{ab} - \xi^c D_c T_{ab} - T_{ac} D_b \xi^c - T_{bc} D_a \xi^c, \quad (2.30)$$

$$\delta T_{ai} \rightarrow \delta T_{ai} - T_{ab} \hat{D}_i \xi^b - r^2 P D_a (\xi_i / r^2), \quad (2.31)$$

$$\delta T_{ij} \rightarrow \delta T_{ij} - \xi^a D_a (r^2 P) \gamma_{ij} - P(\hat{D}_i \xi_j + \hat{D}_j \xi_i). \quad (2.32)$$

We can of course classify the gauge transformations above into the three tensorial types, and discuss each type separately. As is clear from the decomposition theorem (i), the generator ξ^M has only the vector- and scalar-type component; it is decomposed as

$$\xi_a = T_a, \quad \xi_i = V_i + \hat{D}_i S, \quad (2.33)$$

where $\gamma^{ij} \hat{D}_i V_j = 0$. Any tensor-type perturbation variable is by itself gauge-invariant: we have $(h_T^{(2)}{}_{ij}, \delta T_T^{(2)}{}_{ij})$. For the vector type perturbations, the gauge transformation

law of the metric perturbation is given as follows:

$$h_{ai}^{(1)} \rightarrow h_{ai}^{(1)} - r^2 D_a \left(\frac{V_i}{r^2} \right), \quad (2.34)$$

$$h_{Ti}^{(1)} \rightarrow h_{Ti}^{(1)} - V_i. \quad (2.35)$$

Inspecting the transformation laws, we find the following combination is gauge-invariant:

$$F_{ai}^{(1)} = h_{ai}^{(1)} - r^2 D_a \left(\frac{h_{Ti}^{(1)}}{r^2} \right) \quad (2.36)$$

Similarly, inspecting the gauge-transformation law of the matter perturbation, we find two gauge-invariant variables:

$$\tau_{ai}^{(1)} := \delta T_{ai}^{(1)} - P h_{ai}^{(1)}, \quad (2.37)$$

$$\tau_{ij}^{(1)} := 2\hat{D}_{(i} \delta T_{Tj)}^{(1)} - 2P \hat{D}_{(i} h_{Tj)}^{(1)}. \quad (2.38)$$

Any vector-type gauge invariant variable can be expressed as a linear combination of $(F_{ai}^{(1)}, \tau_{ai}^{(1)}, \tau_{ij}^{(1)})$ and their derivatives.

The above equations, (2.36), (2.37), (2.38), defining the gauge invariant variables may be viewed in a way that $(h_{ai}^{(1)}, \delta T_{ai}^{(1)}, \delta T_{Tj}^{(1)})$ are expressed in terms of the gauge-invariants $(F_{ai}^{(1)}, \tau_{ai}^{(1)}, \tau_{ij}^{(1)})$, and $h_{Ti}^{(1)}$. Since, under gauge-transformation, $h_{Ti}^{(1)}$ behaves just like $-\xi_i$, it can be chosen to take any value, and therefore one may view that $h_{Ti}^{(1)}$ alone is responsible for the gauge ambiguity. This, in turn, implies that the specification of $h_{Ti}^{(1)}$ in terms of $(F_{ai}^{(1)}, \tau_{aj}^{(1)}, \tau_{ij}^{(1)})$, corresponds to fixing the gauge freedom of the vector type perturbation.

For the scalar-type metric perturbation, the gauge transformation law is explicitly given as follows:

$$h_{ab} \rightarrow h_{ab} - 2D_{(a} T_{b)}, \quad (2.39)$$

$$h_a \rightarrow h_a - T_a - r^2 D_a \left(\frac{S}{r^2} \right), \quad (2.40)$$

$$h_L \rightarrow h_L - 2r(D^a r)T_a - \frac{2}{n} \hat{\Delta} S, \quad (2.41)$$

$$h_T \rightarrow h_T - 2S. \quad (2.42)$$

Now let us define $X_M = (X_a, X_i = \hat{D}_i X_L)$ by

$$X_a := -h_a + \frac{r^2}{2} D_a \left(\frac{h_T}{r^2} \right), \quad X_L := -\frac{h_T}{2}. \quad (2.43)$$

Then, noting that X_M gauge-transforms as $X_M \rightarrow X_M + \xi_M$, i.e.,

$$(X_a, X_L) \rightarrow (X_a + T_a, X_L + S), \quad (2.44)$$

we can immediately find gauge-invariant combinations:

$$F^{(0)}_{ab} = h_{ab} + 2D_{(a}X_{b)}, \quad (2.45)$$

$$F^{(0)} = h_L + 2r(D^a r)X_a + \frac{2}{n}\hat{\Delta}X_L. \quad (2.46)$$

As for the matter perturbation, we find gauge-invariant combinations:

$$\Sigma^{(0)}_{ab} := \delta T_{ab} + X^c D_c T_{ab} + T_{ac} D_b X^c + T_{bc} D_a X^c, \quad (2.47)$$

$$\Sigma^{(0)}_{ai} := \hat{D}_i \delta T_a + T_{ab} \hat{D}_i X^b + r^2 P D_a \left(\frac{\hat{D}_i X_L}{r^2} \right), \quad (2.48)$$

$$\Sigma^{(0)} := \delta T_L - P h_L + r^2 X^a D_a P, \quad (2.49)$$

$$\Pi^{(0)}_{ij} := \hat{L}_{ij} \delta T_T + 2P \hat{L}_{ij} X_L. \quad (2.50)$$

Any scalar-type gauge invariant variable can be expressed as a linear combination of the gauge-invariant variables $(F^{(0)}_{ab}, F^{(0)}, \Sigma^{(0)}_{ab}, \Sigma^{(0)}_{ai}, \Sigma^{(0)}, \Pi^{(0)}_{ij})$ and their derivatives.

One may view the above defining equations, (2.43), (2.45), (2.46), (2.47), (2.48), (2.49), (2.50), in such a way that all the scalar-type perturbations (h_{ab}, h_a, h_L, h_T) and $(\delta T_{ab}, \delta T_a, \delta T_L, \delta T_T)$ are expressed in terms of $F^{(0)}_{ab}, F^{(0)}, \Sigma^{(0)}_{ab}, \Sigma^{(0)}_{ai}, \Sigma^{(0)}, \Pi^{(0)}_{ij}$ —which are gauge invariant, and X_M —which can be taken as completely arbitrary as the generator ξ_M is. Then, one may say that specifying X_M in terms of the above set of the gauge-invariant variables (or assigning X_M some specific value) corresponds to fixing the gauge freedom. For example, the specification, $X_M = 0$, corresponds to the longitudinal gauge often used in the cosmological context, when $m = 1$, and to the Regge-Wheeler gauge, when $m = 2$.

The perturbed Einstein equations are gauge-invariant and therefore can be written in terms of the gauge-invariant variables introduced above. Note that at this point all variables are functions on \mathcal{M} , being dependent upon the $(m + n)$ -coordinates, x^M . In the next section, we introduce tensor harmonics on \mathcal{K}^n , so that by expanding the perturbation variables in terms of the tensor harmonics, we can separate variables and reduce the relevant equations in \mathcal{M} to equations in \mathcal{N}^m .

As a specific example of a source for gravitational field, let us consider the Maxwell field in the $m = 2$ case. Perturbation of the field strength $\delta \mathcal{F}_{MN}$ satisfies the perturbed Maxwell equations:

$$\nabla_{[M} \delta \mathcal{F}_{NL]} = 0, \quad \delta (\nabla_N \mathcal{F}^{MN}) = J^M, \quad (2.51)$$

where here and in the following the external current J^M is treated as a first-order quantity. The first equation implies that $\delta \mathcal{F}_{MN}$ is expressed in terms of the perturbation of the vector potential, $\delta \mathcal{A}_M$, as $\delta \mathcal{F}_{MN} = 2\nabla_{[M} \delta \mathcal{A}_{N]}$. Therefore $\delta \mathcal{F}_{MN}$ does not contain any tensor-type perturbation. The second equation gives two set of equations,

$$\frac{1}{r^n} D_b (r^n (\delta \mathcal{F})^{ab}) + \hat{D}_i (\delta \mathcal{F})^{ai} + E_0 \epsilon^{ab} \left(\frac{1}{2} D_b (h^i{}_i - h^c{}_c) - \hat{D}_i h^i{}_b \right) = J^a, \quad (2.52a)$$

$$\frac{1}{r^{n-2}} D_a \left[r^{n-2} \left((\delta \mathcal{F})_i{}^a + E_0 \epsilon^{ab} h_{ib} \right) \right] + \hat{D}_j \delta \mathcal{F}_i{}^j = J_i, \quad (2.52b)$$

where note that $(\delta\mathcal{F})^{MN} = g^{ML}g^{NK}\delta\mathcal{F}_{LK}$.

The contribution of electromagnetic field perturbations to the energy-momentum tensor are given by

$$\delta T_{ab}^{(\text{em})} = \frac{E_0}{2} \left(\epsilon^{cd} \delta\mathcal{F}_{cd} + E_0 h^c{}_c \right) g_{ab} - \frac{1}{2} E_0^2 h_{ab}, \quad (2.53a)$$

$$\delta T^{(\text{em})a}{}_i = -E_0 \epsilon^{ab} \delta\mathcal{F}_{bi}, \quad (2.53b)$$

$$\delta T^{(\text{em})i}{}_j = -\frac{E_0}{2} \left(\epsilon^{cd} \delta\mathcal{F}_{cd} + E_0 h^c{}_c \right) \delta^i{}_j. \quad (2.53c)$$

For vector-type perturbations, we note that $\delta\mathcal{F}_{MN}$ is gauge-invariant as well as invariant under a coordinate gauge transformation $x^M \rightarrow x^M + \xi^M$. We find

$$\delta\tau_{ai}^{(\text{em})(1)} = -E_0 \epsilon_{ab} D^b \delta\mathcal{A}_i^{(1)}, \quad \delta\tau_{ij}^{(\text{em})(1)} = 0, \quad (2.54)$$

where $\hat{D}^i \delta\mathcal{A}_i^{(1)} = 0$. We find from the first of Eqs. (2.52b) that a vector perturbation of the Maxwell field can be expressed in terms of the gauge-invariant variable $\delta\mathcal{A}_i^{(1)}$ on \mathcal{N}^2 as,

$$\delta\mathcal{F}_{ab} = 0, \quad \delta\mathcal{F}_{ai} = D_a \delta\mathcal{A}_i^{(1)}, \quad \delta\mathcal{F}_{ij} = \hat{D}_i \delta\mathcal{A}_j^{(1)} - \hat{D}_j \delta\mathcal{A}_i^{(1)}. \quad (2.55)$$

For scalar-type perturbations, $\delta\mathcal{F}_{MN}$ gauge-transforms as:

$$\delta\mathcal{F}_{ab} \rightarrow \delta\mathcal{F}_{ab} - D_c(E_0 \xi^c) \epsilon_{ab}, \quad \delta\mathcal{F}_{aj} \rightarrow \delta\mathcal{F}_{aj} - E_0 \epsilon_{ab} \hat{D}_j \xi^b. \quad (2.56)$$

Note that $\delta\mathcal{F}_{ij} = 0$ for a scalar perturbation. So, we can immediately find gauge-invariant combinations, $\mathcal{E}^{(0)}$, $\mathcal{E}_a^{(0)}$, given by

$$\epsilon_{ab} \mathcal{E}^{(0)} = \delta\mathcal{F}_{ab} + \epsilon_{ab} D_c(E_0 X^c), \quad (2.57)$$

$$\epsilon_{ab} \hat{D}_i \mathcal{E}^{(0)b} = \delta\mathcal{F}_{ai} + E_0 \epsilon_{ab} \hat{D}_i X^b. \quad (2.58)$$

The gauge-invariant variables introduced above are then written as

$$\Sigma^{(\text{em})(0)}{}_{ab} = -\frac{E_0^2}{2} \left(F^{(0)}{}_{ab} - F^{(0)c}{}_c g_{ab} \right) - E_0 \mathcal{E}^{(0)} g_{ab}, \quad (2.59)$$

$$\Sigma^{(\text{em})(0)}{}_{ai} = -E_0 \hat{D}_i \mathcal{E}_a^{(0)}, \quad (2.60)$$

$$\Sigma^{(\text{em})(0)} = r^2 \left(E_0 \mathcal{E}^{(0)} - \frac{E_0^2}{2} F^{(0)c}{}_c \right), \quad (2.61)$$

$$\Pi^{(\text{em})(0)}{}_{ij} = 0. \quad (2.62)$$

§3. Harmonic tensors on the Einstein space \mathcal{K}^n

When we write down perturbation equations and solve them, it is often more convenient to expand perturbation variables into Fourier-type harmonic components in terms of harmonic tensors appropriate for each tensorial type on the internal space \mathcal{K}^n . In this section, we summarize the definitions and basic properties of such harmonic tensors relevant to the descriptions in the present paper.

3.1. Scalar harmonics

3.1.1. Definition and properties

Covariant derivative \hat{D} along \mathcal{K}^n appears only in the form of the Laplace-Beltrami operator in the perturbation equations for scalar-type variables because γ_{ij} is the only non-trivial symmetric tensor on the Einstein space \mathcal{K}^n . Hence, the scalar-type perturbation equations reduce to a set of PDEs on \mathcal{N} by expanding scalar-type perturbation variables in terms of scalar harmonic functions on \mathcal{K}^n that satisfy

$$(\hat{\Delta} + k^2)\mathbb{S} = 0. \quad (3.1)$$

Here, when \mathcal{K}^n is non-compact, we assume that $-\hat{\Delta}$ is extended to a non-negative self-adjoint operator in the L^2 -space of functions on \mathcal{K}^n . Hence, $k^2 \geq 0$. Such an extension is unique if \mathcal{K}^n is complete²⁰⁾ and is given by the Friedrichs self-adjoint extension of the symmetric and non-negative operator $-\hat{\Delta}$ on $C_0^\infty(\mathcal{K}^n)$.

If \mathcal{K}^n is closed, the spectrum of $\hat{\Delta}$ is completely discrete, each eigenvalue has a finite multiplicity, and the lowest eigenvalue is $k^2=0$, whose eigenfunction is a constant. A perturbation corresponding to such a constant mode generally represents a variation of the parameters of the background solution such as λ , M , and Q . Thus, this mode is relevant to arguments on perturbative uniqueness of a background solution. Note that when $k^2 = 0$ is contained in the full spectrum but does not belong to the point spectrum, as in the case $\mathcal{K}^n = \mathbb{R}^n$, it can be ignored without loss of generality.

For modes with $k^2 > 0$, we can use the vector fields and the symmetric trace-free tensor fields defined by

$$\mathbb{S}_i = -\frac{1}{k}\hat{D}_i\mathbb{S}, \quad (3.2a)$$

$$\mathbb{S}_{ij} = \frac{1}{k^2}\hat{D}_i\hat{D}_j\mathbb{S} + \frac{1}{n}\gamma_{ij}\mathbb{S}; \quad \mathbb{S}^i{}_i = 0 \quad (3.2b)$$

to expand vector and symmetric trace-free tensor fields, respectively. Note that \mathbb{S}_i is also an eigenmode of the operator $\hat{D} \cdot \hat{D}$, i.e.,

$$[\hat{D} \cdot \hat{D} + k^2 - (n-1)K]\mathbb{S}_i = 0, \quad (3.3)$$

while \mathbb{S}_{ij} satisfies

$$(\hat{\Delta}_L - k^2)\mathbb{S}_{ij} = 0, \quad (3.4)$$

where $\hat{\Delta}_L$ is the Lichnerowicz operator defined by

$$\hat{\Delta}_L h_{ij} := -\hat{D} \cdot \hat{D} h_{ij} - 2\hat{R}_{ikjl}h^{kl} + 2(n-1)Kh_{ij}. \quad (3.5)$$

When \mathcal{K} is a constant curvature space, this operator is related to the Laplace-Beltrami operator by

$$\hat{\Delta}_L = -\hat{\Delta} + 2nK, \quad (3.6)$$

In the case of scalar harmonics, the modes with $k^2 = nK$ are exceptional. Given our assumption, these modes exist only for $K = 1$. Because \mathcal{K}^n is compact and

closed in this case, from the identity

$$\hat{D}_j \mathbb{S}^j_i = \frac{n-1}{n} \frac{k^2 - nK}{k} \mathbb{S}_i, \quad (3.7)$$

we have $\hat{D}_j \mathbb{S}^j_i = 0$. From this, it follows that $\int d^n z \sqrt{\gamma} \mathbb{S}_{ij}^* \mathbb{S}^{ij} = 0$. Hence, \mathbb{S}_{ij} vanishes identically.

We can further show that the second smallest eigenvalue for $-\hat{\Delta}$ is equal to or greater than nK when $K > 0$. To see this, let us define Q_{ij} by

$$Q_{ij} := \hat{L}_{ij} Y = \hat{D}_i \hat{D}_j Y - \frac{1}{n} \gamma_{ij} \hat{\Delta} Y.$$

Then, we have the identity

$$Q_{ij} Q^{ij} = \hat{D}^i (\hat{D}^i Y \hat{D}_i \hat{D}_j Y - Y \hat{D}_i \hat{\Delta} Y - \hat{R}_{ij} \hat{D}^j Y) + Y \left[\hat{\Delta} (\hat{\Delta} + (n-1)K) \right] Y - \frac{1}{n} (\hat{\Delta} Y)^2.$$

For $Y = \mathbb{S}$, integrating this identity, we obtain the constraint on the second eigenvalue

$$k^2 \geq nK. \quad (3.8)$$

For $\mathcal{K}^n = S^n$, the equality holds for the second smallest eigenvalue as we see soon.

3.1.2. Harmonic functions on S^n

Explicit expressions for the harmonic functions on higher-dimensional spaces sometimes become necessary to investigate the global structure of perturbations. The multiplicity of eigenvalues also has a crucial importance in applying the perturbation theory to black hole evaporation. Here, we give such information for harmonic functions on S^n .

There are several ways to express higher-dimensional spherical harmonic functions. For example, we can get expressions in terms of special functions by solving the recurrence relation obtained by dimensional reduction. In this paper, we give a different approach in which harmonic functions are expressed in terms of the homogeneous cartesian coordinates for the Euclidean space E^{n+1} containing the unit sphere S^n .

Let us denote the homogeneous cartesian coordinates of S^n by Ω^A ($A = 1, \dots, n+1$); $\Omega \cdot \Omega = 1$. Then, we can show that Ω^A satisfies

$$\hat{D}_i \hat{D}_j \Omega^A = -\gamma_{ij} \Omega^A, \quad (3.9a)$$

$$\hat{\Delta} \hat{D}_i \Omega^A = -\hat{D}_i \Omega^A, \quad (3.9b)$$

$$\hat{D}_i \Omega^A \hat{D}^i \Omega^B = \delta^{AB} - \Omega^A \Omega^B. \quad (3.9c)$$

From these formulae, the following theorem holds.

Theorem 3.1 (Scalar harmonics on S^n). Let us define the function $Y_{\mathbf{a}}$ on S^n by

$$Y_{\mathbf{a}}(\Omega) = a_{A_1 \dots A_\ell} \Omega^{A_1} \dots \Omega^{A_\ell} \quad (3.10)$$

in terms of a constant tensor $\mathbf{a} = (a_{A_1 \dots A_\ell})$ ($A_1, \dots, A_\ell = 1, \dots, n+1$). Then, the following statements hold:

1) $Y_{\mathbf{a}}$ is a harmonic function on S^n with the eigenvalue

$$k^2 = \ell(\ell + n - 1), \quad \ell = 0, 1, 2, \dots, \quad (3.11)$$

if and only if \mathbf{a} satisfies the conditions

$$a_{A_1 \dots A_\ell} = a_{(A_1 \dots A_\ell)}, \quad (3.12a)$$

$$a_{A_1 \dots A_{\ell-2}}{}^B{}_B = 0 \quad (\ell \geq 2). \quad (3.12b)$$

2) The harmonic functions $\{Y_{\mathbf{a}}\}$ form a complete basis in $L^2(S^2)$. Two harmonic functions with different values of ℓ are orthogonal and those with the same ℓ have the inner product

$$(Y_{\mathbf{a}}, Y_{\mathbf{a}'}) := \int d^n \Omega \bar{Y}_{\mathbf{a}} Y_{\mathbf{a}'} = C(n, \ell) \bar{a}^{j_1 \dots j_\ell} a'_{j_1 \dots j_\ell}, \quad (3.13a)$$

$$C(n, \ell) = \frac{2\pi^{\frac{n+1}{2}} \ell!}{2^\ell \Gamma(\frac{n+1}{2} + \ell)}. \quad (3.13b)$$

3) The multiplicity of the ℓ -eigenvalue is given by

$$N_S^\ell(S^n) = \frac{(n + 2\ell - 1)(n + \ell - 2)!}{(n - 1)! \ell!}. \quad (3.14)$$

Proof. The first statement follows from

$$\begin{aligned} \hat{\Delta}_n Y_{\mathbf{a}} &= -\ell n Y_{\mathbf{a}} + \sum_{p \neq q} a_{A_1 \dots A_\ell} \Omega^{A_1} \dots \hat{D}_k \Omega^{A_p} \dots \hat{D}^k \Omega^{A_q} \dots \Omega^{A_\ell} \\ &= -\ell(\ell + n - 1) Y_{\mathbf{a}} + \frac{\ell(\ell - 1)}{2} a_{A_1 \dots A_{\ell-2}}{}^B{}_B \Omega^{A_1} \dots \Omega^{A_{\ell-2}} \end{aligned} \quad (3.15)$$

which can be easily verified with the helps of the identities, Eqs. (3.9).

Next, all polynomials in the cartesian coordinates x^A for E^{n+1} are dense in the space of continuous functions in the unit cube in E^{n+1} , hence its restriction on S^n is also dense in the space of continuous functions on S^n . This implies that all the harmonic functions of the type $Y_{\mathbf{a}}$ are dense in the function space $L^2(S^n)$. This proves the completeness. The inner product of these harmonic functions can be calculated as follows. First, by differentiating the function

$$F(r^2) := \int d^n \Omega e^{i\mathbf{r} \cdot \Omega} = \Omega_{n-1} \int_0^\pi d\theta \sin^{n-1} \theta e^{i r \cos \theta} = \Omega_n \frac{\Gamma(\frac{n+1}{2})}{(r/2)^{(n-1)/2}} J_{\frac{n-1}{2}}(r) \quad (3.16)$$

repeatedly with respect to x^A ($\mathbf{r} = (x^A)$), we obtain

$$\int d^n \Omega Y_{\mathbf{a}}(\Omega) e^{i\mathbf{r} \cdot \Omega} = (-1)^\ell 2^\ell x^{A_1} \dots x^{A_\ell} a_{A_1 \dots A_\ell} F^{(\ell)}(r^2). \quad (3.17)$$

Differentiating this ℓ -th times with respect to x^A and putting $x^A = 0$ yield Eq. (3.13). The multiplicity formula of the eigenvalue can be easily obtained by just counting the linearly independent solutions to Eq. (3.12). \square

3.2. Vector harmonics

3.2.1. Definition and properties

A harmonic vector is defined as a vector field on S^n satisfying

$$(\hat{D} \cdot \hat{D} + k_v^2) \mathbb{V}_i = 0; \quad \hat{D}^i \mathbb{V}_i = 0. \quad (3.18)$$

From this we can define a symmetric trace-free tensor of rank 2 by

$$\mathbb{V}_{ij} = -\frac{1}{k_v} \hat{D}_{(i} \mathbb{V}_{j)}, \quad (3.19)$$

where the factor $1/k_v$ is just a convention (see below for the case $k_v = 0$). This tensor is an eigentensor of the Lichnerowicz operator,

$$\hat{\Delta}_L \mathbb{V}_{ij} = [k_v^2 + (n-1)K] \mathbb{V}_{ij}, \quad (3.20)$$

but is not an eigentensor of the Laplacian in general when \mathcal{K}^n is not a constant curvature space.

In this paper, we assume that the Laplacian $-\hat{D} \cdot \hat{D}$ is extended to a non-negative self-adjoint operator in the L^2 -space of divergence-free vector fields on \mathcal{K}^n , in order to guarantee the completeness of the vector harmonics. Because $-\hat{D} \cdot \hat{D}$ is symmetric and non-negative in the space consisting of smooth divergence-free vector fields with compact support, it always possesses a Friedrichs extension that has the desired property.²¹⁾ With this assumption, k_v^2 is non-negative.

One subtlety that arises in this harmonic expansion concerns the zero modes of the Laplacian. If \mathcal{K}^n is closed, from the integration of the identity $\hat{D}^i (V^j \hat{D}_i V_j) - \hat{D}^i V^j \hat{D}_i V_j = V^j \hat{D} \cdot \hat{D} V_j$, it follows that $\hat{D}_i V_j = 0$ for $k_v^2 = 0$. Hence, we cannot construct a harmonic tensor from such a vector harmonic. We obtain the same result even in the case in which \mathcal{K}^n is open if we require that $\mathbb{V}^j \hat{D}_i V_j$ fall off sufficiently rapidly at infinity. In the present paper, we assume that this fall-off condition is satisfied. From the identity $\hat{D}^j \hat{D}_i V_j = \hat{D}_i \hat{D}^j V_j + (n-1)K \hat{V}_i$, such a zero mode exists only in the case $K = 0$. We can further show that vector fields satisfying $\hat{D}_i V_j = 0$ exist if and only if \mathcal{K}^n is a product of a locally flat space and an Einstein manifold with vanishing Ricci tensor.

More generally, \mathbb{V}_{ij} vanishes if \mathbb{V}_i is a Killing vector. In this case, from the relation

$$2k_v \hat{D}_j \mathbb{V}^j_i = [k_v^2 - (n-1)K] \mathbb{V}_i, \quad (3.21)$$

k_v^2 takes the special value $k_v^2 = (n-1)K$. Because $k_v^2 \geq 0$, this occurs only for $K = 0$ or $K = 1$. In the case $K = 0$, this mode corresponds to the zero mode discussed above. In the case $K = 1$, since we are assuming that \mathcal{K}^n is complete, \mathcal{K}^n is compact and closed, as known from Myers’ theorem,²²⁾ and we can show the converse, i.e. that if $k_v^2 = (n-1)K$, then \mathbb{V}_{ij} vanishes, by integrating the identity $0 = \mathbb{V}^i \hat{D}^j \mathbb{V}_{ij} = \hat{D}^j (\mathbb{V}^i \mathbb{V}_{ij}) + k_v \mathbb{V}^{ij} \mathbb{V}_{ij}$ over \mathcal{K}^n . Furthermore, using the same identities

$$2D_{[i} V_{j]} D^{[i} V^{j]} = 2D_i (V_j D^{[i} V^{j]}) + V_j [-\Delta + (n-1)K] V^j, \quad (3.22a)$$

$$2D_{(i} V_{j)} D^{(i} V^{j)} = 2D_i (V_j D^{(i} V^{j)}) + V_j [-\Delta - (n-1)K] V^j. \quad (3.22b)$$

we can show that there is no eigenvalue in the range $0 \leq k_v^2 \leq (n-1)|K|$ where the second equality holds only for $K = -1$.

3.2.2. Harmonic vectors on S^n

We can give explicit expressions for vector harmonics on S^n in terms of the homogeneous coordinates using Theorem 3.1.

Theorem 3.2 (Harmonic vectors on S^n). Let us define the vector field $V_{\mathbf{b}}^i$ by

$$V_{\mathbf{b}}^i = b_{A_1 \dots A_\ell; B} \Omega^{A_1} \dots \Omega^{A_\ell} \hat{D}^i \Omega^B. \quad (3.23)$$

in terms of a constant tensor $\mathbf{b} = (a_{A_1 \dots A_\ell; B})(A_1, \dots, A_\ell, B = 1, \dots, n+1)$. Then, the following statements hold:

- 1) $V_{\mathbf{b}}^i$ is a divergence-free harmonic vector on S^n with eigenvalue

$$k_v^2 = \ell(\ell + n - 1) - 1, \quad \ell = 1, 2, \dots, \quad (3.24)$$

if and only if the constant tensor \mathbf{b} satisfies the conditions

$$b_{A_1 \dots A_\ell; B} = b_{(A_1 \dots A_\ell); B}, \quad (3.25a)$$

$$b_{A_1 \dots A_{\ell-2} i; B} = 0, \quad (3.25b)$$

$$b_{(A_1 \dots A_\ell; A_{\ell+1})} = 0. \quad (3.25c)$$

- 2) All harmonic vectors $\{V_{\mathbf{b}}^i\}$ form a complete basis in the L^2 space of divergence-free vector fields on S^n . Two harmonic vectors with different values of ℓ are orthogonal and those with the same ℓ have the inner product

$$(V_{\mathbf{b}}, V_{\mathbf{b}'}) := \int d^n \Omega (\bar{V}_{\mathbf{b}})_i V_{\mathbf{b}'}^i = C(n, \ell) \bar{\mathbf{b}} \cdot \mathbf{b}', \quad (3.26)$$

where $C(n, \ell)$ is the number given in Eq. (3.13)D

- 3) The multiplicity of the ℓ -th eigenvalue is

$$N_V^\ell(S^n) = \frac{(n+2\ell-1)(n+\ell-1)(n+\ell-3)!}{(\ell+1)(\ell-1)!(n-2)!}. \quad (3.27)$$

Proof. For a harmonic vector V^i with the eigenvalue k_v^2 , let us define a set of functions on S^n by $V^B = \hat{V}^i \hat{D}_i \Omega^B$. Then, from Eq. (3.9) we obtain $\hat{\Delta} V^B = -(k_v^2 + 1)V^B$. This implies that V^B is a harmonic function with the eigenvalue $k^2 = k_v^2 + 1 = \ell(\ell + n - 1)$. Therefore, $V_i = V_B \hat{D}_i \Omega^B$ can be written in terms of a constant tensor $b_{A_1 \dots A_\ell; B}$ satisfying the conditions, Eqs. (3.25a) and (3.25b) as in Eq. (3.23). The divergence of this expression can be written

$$\hat{D}_i V^i = [-(n+\ell)b_{A_1 \dots A_\ell; A_{\ell+1}} - \ell a_{A_1 \dots A_{\ell-1}} \delta_{A_\ell A_{\ell+1}}] \Omega^{A_1} \dots \Omega^{A_{\ell+1}}, \quad (3.28)$$

where

$$a_{A_1 \dots A_{\ell-1}} := b_{A_1 \dots A_{\ell-1} i; i}. \quad (3.29)$$

From this, it follows that the divergence free condition for V^i can be expressed as

$$(n+\ell)b_{(A_1 \dots A_\ell; A_{\ell+1})} = \ell a_{(A_1 \dots A_{\ell-1}} \delta_{A_\ell A_{\ell+1})}.$$

After a short calculation, we find that this condition is equivalent to Eq. (3.25c). This proves the first statement.

The completeness in the second statement immediately follows from the completeness of the harmonic vectors. The formula for the inner product can be derived by the same method as used for the harmonic scalars.

Finally, we can show that the conditions, Eqs. (3.25), on \mathbf{b} are reduced to the conditions on $b_{A'_1 \dots A'_\ell; B'}$ and $b_{n+1 A'_1 \dots A'_{\ell-1}; B'}$ with $A'_1, \dots, A'_\ell, B' = 1, \dots, n$,

$$b_{(A'_1 \dots A'_\ell; B')} = 0, \quad (3.30a)$$

$$(\ell - 2)b_{n+1}{}^{B'}{}_{B'(A'_1 \dots A'_{\ell-3}; A'_{\ell-2})} + 3b_{n+1 A'_1 \dots A'_{\ell-3}}{}^{B'}{}_{B'}, \quad (3.30b)$$

and all the other components are uniquely determined from these components. By counting the number of linealy independent solutions to these conditions, we obtain the multiplicity in the theorem. \square

3.3. Tensor harmonics

3.3.1. Definition and properties

In the Einstein space \mathcal{K}^n , no condition is imposed directly on the Riemann tensor $R^i{}_{jkl}$ itself, though the Ricci tensor is assumed to be proportional to the metric. Hence, in general the Lichnerowicz operator appears in the tensor-type perturbation equation instead of a simple sum of the Laplace-Beltrami operator and scalars as \mathcal{K}^n -dependent part. Thus, we have to use the eigentensors to the Lichnerowicz operators to expand tensor-type perturbations on a generic Einstein space \mathcal{K}^n :

$$\hat{\Delta}_L \mathbb{T}_{ij} = \lambda_L \mathbb{T}_{ij}, \quad (3.31)$$

$$\mathbb{T}^i{}_i = 0, \quad \hat{D}^j \mathbb{T}_{ij} = 0. \quad (3.32)$$

Note that the Lichnerowicz operator preserves the trace-free and transverse conditions. When \mathcal{K}^n is a constant curvature space, \mathbb{T}_{ij} becomes a harmonic tensor on \mathcal{K}^n ,

$$(\hat{\Delta} + k_t^2) \mathbb{T}_{ij} = 0, \quad k_t^2 = \lambda_L - 2nK. \quad (3.33)$$

Note also that an Einstein space with dimension equal to or smaller than 3 is always a constant curvature space. Further, there exist no symmetric harmonic tensor with rank 2 on S^2 and very special ones on T^2 and H^2/Γ . To be precise, the following theorem holds:

Theorem 3.3 (Harmonic tensors on 2-dimensional constant curvature space). Let \mathcal{K} be a two-dimensional closed surface with a constant curvature K . Then, a symmetric harmonic tensor \mathbb{T}_{ij} with rank 2 represents a moduli deformation of \mathcal{K} and exists only when $K \leq 0$. For T^2 ($K = 0$), \mathbb{T}_{ij} is a constant trace-free tensor with $k^2 = 0$ in the chart in which $ds^2 = dx^2 + dy^2$, while $k^2 = -2K$ for H^2/Γ ($K < 0$).

Very little is known about the spectrum of the Lichnerowicz operator on a generic Einstein space. However, we can easily show that

$$k_t^2 \geq n|K|, \quad (3.34)$$

for tensor harmonics on a constant curvature space with the helps of the identities

$$2D_{[i}T_{j]k}D^{[i}T^{j]k} = 2D^i(T_{jk}D^{[i}T^{j]k}) + T_{jk}(-\Delta + nK)T^{jk}, \quad (3.35a)$$

$$2D_{(i}T_{j)k}D^{(i}T^{j)k} = 2D^i(T_{jk}D^{(i}T^{j)k}) + T_{jk}(-\Delta - nK)T^{jk}. \quad (3.35b)$$

3.3.2. Harmonic tensors on S^n

We can give explicit expressions for the symmetric harmonic tensors of rank 2 on S^n in terms of homogeneous coordinates as in the case of harmonic vectors.

Theorem 3.4 (Harmonic tensors on S^n). Let us define a tensor of rank 2 by

$$T_{\mathbf{c}ij} = c_{A_1 \dots A_\ell; B_1 B_2} \Omega^{A_1} \dots \Omega^{A_\ell} \hat{D}_i \Omega^{B_1} \hat{D}_j \Omega^{B_2} \quad (3.36)$$

where $\mathbf{c} = (c_{A_1 \dots A_\ell; B_1 B_2})$ ($A_1, \dots, A_\ell, B_1, B_2 = 1, \dots, n+1$) is a constant tensor on E^{n+1} . Then, the following statements hold:

- i) $T_{\mathbf{c}ij}$ is a harmonic tensor with the eigenvalue

$$k^2 = \ell(\ell + n - 1) - 2 \quad (\ell = 2, \dots) \quad (3.37)$$

if and only if \mathbf{c} satisfies the following conditions:

$$c_{A_1 \dots A_\ell; B_1 B_2} = c_{(A_1 \dots A_\ell); B_1 B_2}, \quad (3.38a)$$

$$c_{A_1 \dots A_{\ell-2} A^A; B_1 B_2} = 0, \quad (3.38b)$$

$$c_{(A_1 \dots A_\ell; A_{\ell+1})}^B = c_{(A_1 \dots A_\ell; A_{\ell+1})}^B = 0. \quad (3.38c)$$

- 2) The set of all harmonic tensors of the form $T_{\mathbf{c}ij}$ forms a complete basis for the L^2 space of trace-free and divergence-free tensors of rank 2 on S^n . Two such harmonic tensors with different values of ℓ are orthogonal and those with the same ℓ have the inner product

$$(T_{\mathbf{c}}, T_{\mathbf{c}'}) := \int d^n \Omega \bar{T}_{\mathbf{c}ij} T_{\mathbf{c}'}^{ij} = C(n, \ell) \bar{\mathbf{c}} \cdot \mathbf{c}', \quad (3.39)$$

where $C(n, \ell)$ is the same constant as that in Eq. (3.13)D

- 3) For $\ell \geq 2, n \geq 2$, the multiplicity of the ℓ -th eigenvalue for symmetric harmonic tensors of rank 2 is

$$N_T^\ell(S^n) = \frac{(n+1)(n-2)(n+\ell)(n+2\ell-1)(n+\ell-3)!}{2\ell(\ell-2)!(n-1)!}. \quad (3.40)$$

Proof. The first two statements can be easily proved by methods similar to those for the harmonic vector. To prove the last statement, lengthy calculations or sophisticated considerations based on group representation theory are required. See Ref. 23) for details. \square

§4. Tensor-type perturbations

4.1. Generic background

We first consider tensor perturbations in an $(m+n)$ -dimensional generic background metric, Eq. (2.1), and the energy-momentum tensor, Eq. (2.5). We have

already seen in the previous section that $h_T^{(2)}{}_{ij}$, $\delta T_T^{(2)}{}_{ij}$ are by themselves gauge-invariant. We expand these two in terms of the eigentensors \mathbb{T}_{ij} of the Lichnerowicz operator $\hat{\Delta}_L$ introduced in the previous section as follows:

$$h_T^{(2)}{}_{ij} = 2r^2 H_T \mathbb{T}_{ij}, \quad \delta T_T^{(2)}{}_{ij} = r^2 (\tau_T + 2PH_T) \mathbb{T}_{ij}. \quad (4.1)$$

The Einstein equations in terms of the gauge-invariant coefficients H_T and τ_T are obtained from Eq. (23) in Ref. 24) with the replacement $k^2 \rightarrow \lambda_L - 2nK$, where λ_L is the eigenvalue of the Lichnerowicz operator. The result is expressed by the single equation

$$\square H_T + \frac{n}{r} Dr \cdot DH_T - \frac{\lambda_L - 2(n-1)K}{r^2} H_T = -\kappa^2 \tau_T. \quad (4.2)$$

We emphasise that this equation holds whenever the background metric is given in the form of Eq. (2.1), irrespective to the dimension of \mathcal{N}^m , and therefore applies to a more general background than that of a static black hole. For example, it has been applied to the stability analysis²⁵⁾ of Myers-Perry black holes with a single rotation, whose metric takes the warped product form of Eq. (2.1), with \mathcal{K}^n being the $(d-4)$ -dimensional unit sphere.

We also note that Eq. (4.2) is precisely the same form as the equation of motion for a massless test scalar field on the same background spacetime if H_T is viewed as the scalar field with the angular momentum number being $\lambda_L - 2(n-1)K$.

4.2. Static black hole background

We turn to the black hole background, Eq. (2.14), for which $m = 2$. If one introduces the master variable Φ by

$$\Phi = r^{n/2} H_T, \quad (4.3)$$

Eq. (4.2) can be put into the canonical form

$$\square \Phi - \frac{V_T}{f} \Phi = -\kappa^2 r^{n/2} \tau_T, \quad (4.4)$$

where

$$V_T = \frac{f}{r^2} \left[\lambda_L - 2(n-1)K + \frac{nr f'}{2} + \frac{n(n-2)f}{4} \right]. \quad (4.5)$$

In particular, for $f(r)$ given by Eq. (2.15), V_T is expressed as

$$V_T = \frac{f}{r^2} \left[\lambda_L + \frac{n^2 - 10n + 8}{4} K - \frac{n(n+2)}{4} \lambda r^2 + \frac{n^2 M}{2r^{n-1}} - \frac{n(3n-2)Q^2}{4r^{2n-2}} \right]. \quad (4.6)$$

Note that since an electromagnetic field \mathcal{F}_{ab} is described by a vector field, it does not have any tensor-type component. The electromagnetic field enter the equations for a tensor perturbation only through their effect on the background geometry.

§5. Vector-type perturbations

5.1. General background case

We have already introduced a basis of vector-type gauge-invariant variables $(F_{ai}^{(1)}, \tau_{ai}^{(1)}, \tau_{ij}^{(1)})$ in generic background of $(m+n)$ -dimensions. We expand these gauge invariants in terms of vector harmonics \mathbb{V}_i and write the perturbed Einstein equations for the expansion coefficients.

For generic modes $m_V := k_v^2 - (n-1)K \neq 0$,

$$F_{ai}^{(1)} = rF_a \mathbb{V}_i, \quad \tau_{ai}^{(1)} = r\tau_a \mathbb{V}_i, \quad \tau_{ij}^{(1)} = r^2 \tau_T \mathbb{V}_{ij}, \quad (5.1)$$

and the Einstein equations reduce to

$$\frac{1}{r^{n+1}} D^b \left\{ r^{n+2} \left[D_b \left(\frac{F_a}{r} \right) - D_a \left(\frac{F_b}{r} \right) \right] \right\} - \frac{m_V}{r^2} F_a = -2\kappa^2 \tau_a, \quad (5.2)$$

$$\frac{k}{r^n} D_a (r^{n-1} F^a) = -\kappa^2 \tau_T. \quad (5.3)$$

For the exceptional mode $m_V = 0$, only the following combination

$$F_{ab}^{(1)} = r D_a \left(\frac{F_b}{r} \right) - r D_b \left(\frac{F_a}{r} \right), \quad (5.4)$$

is gauge-invariant. For this mode we have only a single equation

$$\frac{1}{r^{n+1}} D^b (r^{n+1} F_{ab}^{(1)}) = -2\kappa^2 \tau_a. \quad (5.5)$$

5.2. Static black hole background case

In the black hole case with $m = 2$, using the 2-dimensional Levi-Civita tensor ϵ_{ab} , we can rewrite the perturbed Einstein equations as

$$D_a \left(r^{n+1} F^{(1)} \right) - m_V r^{n-1} \epsilon_{ab} F^b = -2\kappa^2 r^{n+1} \epsilon_{ab} \tau^b, \quad (5.6a)$$

$$k_v D_a (r^{n-1} F^a) = -\kappa^2 r^n \tau_T, \quad (5.6b)$$

where

$$F^{(1)} = \epsilon^{ab} r D_a \left(\frac{F_b}{r} \right), \quad (5.7)$$

and F_a is defined by the first of Eq. (5.1) with $F_{ai}^{(1)} \rightarrow h_{ai}^{(1)}$. This should be supplemented by the perturbation of the energy-momentum conservation law

$$D_a (r^{n+1} \tau^a) + \frac{m_V}{2k_v} r^n \tau_T = 0. \quad (5.8)$$

Note that, for $m_V = 0$, the perturbation variables $h_T^{(1)}{}_i$ and $\delta T_T^{(1)}{}_i$ —hence H_T and τ_T —do not exist. The matter variable τ_a is still gauge-invariant, but concerning the metric variables, only the combination $F^{(1)}$ defined in Eq. (5.7) is gauge invariant.

In this case, the Einstein equations are reduced to the single equation (5.6a), and the energy-momentum conservation law is given by Eq. (5.8) without the τ_T term.

These gauge-invariant perturbation equations can be reduced to a single wave equation with a source in the 2-dimensional spacetime \mathcal{N}^2 . First, for the generic modes $m_V \neq 0$, from Eqs. (5.8) and (5.6b) we find that F^a can be written in terms of a variable $\tilde{\Omega}$ as

$$\epsilon^{ab} D_b \tilde{\Omega} = r^{n-1} F^a - \frac{2\kappa^2}{m_V} r^{n+1} \tau^a. \quad (5.9)$$

Inserting this expression into Eq. (5.6a), we obtain the master equation

$$r^n D_a \left(\frac{1}{r^n} D^a \tilde{\Omega} \right) - \frac{m_V}{r^2} \tilde{\Omega} = -\frac{2\kappa^2}{m_V} r^n \epsilon^{ab} D_a (r \tau_b). \quad (5.10)$$

For the special modes $m_V = 0$, it follows from Eq. (5.8) with $\tau_T = 0$ that τ_a can be expressed in terms of a function $\tau^{(1)}$ as

$$r^{n+1} \tau_a = \epsilon_{ab} D^b \tau^{(1)}. \quad (5.11)$$

Inserting this expression into Eq. (5.6a) with $\epsilon^{cd} D_c (F_d/r)$ replaced by $F^{(1)}/r$, we obtain

$$D_a (r^{n+1} F^{(1)}) = -2\kappa^2 D_a \tau^{(1)}. \quad (5.12)$$

Taking into account of the freedom of adding a constant in the definition of $\tau^{(1)}$, we have the general solution

$$F^{(1)} = -\frac{2\kappa^2 \tau^{(1)}}{r^{n+1}}. \quad (5.13)$$

Hence, there exists no dynamical freedom in these special modes. In particular, in the source-free case in which $\tau^{(1)}$ is a constant and $K = 1$, this solution corresponds to adding a small rotation to the background solution.

5.3. Static black hole in Einstein-Maxwell system

5.3.1. Perturbation of electromagnetic fields

For vector-type perturbations, we note that $\delta \mathcal{F}_{MN}$ is gauge-invariant as well as invariant under a coordinate gauge transformation. As shown in Eq. (2.55), a vector perturbation of the Maxwell field can be expressed, in terms of the single gauge-invariant variable \mathcal{A} defined by $\delta \mathcal{A}_i^{(1)} = \mathcal{A} \mathbb{V}_i$ on \mathcal{N}^2 , as

$$\delta \mathcal{F}_{ab} = 0, \quad \delta \mathcal{F}_{ai} = D_a \mathcal{A} \mathbb{V}_i, \quad \delta \mathcal{F}_{ij} = \mathcal{A} \left(\hat{D}_i \mathbb{V}_j - \hat{D}_j \mathbb{V}_i \right). \quad (5.14)$$

We also expand the current J_i as

$$J_i = J \mathbb{V}_i. \quad (5.15)$$

Then we obtain from the second of Eq. (2.52b) the gauge-invariant form for the Maxwell equation,

$$\frac{1}{r^{n-2}} D_a (r^{n-2} D^a \mathcal{A}) - \frac{k_v^2 + (n-1)K}{r^2} \mathcal{A} = -J + r E_0 F^{(1)}. \quad (5.16)$$

In order to complete the formulation of the basic perturbation equations, we must separate the contribution of the electromagnetic field to the source term in the Einstein equation (5.10). The contributions of the electromagnetic field to τ_a and τ_T are given in terms of \mathcal{A} by

$$\tau_a^{(\text{em})} = -\frac{E_0}{r}\epsilon_{ab}D^b\mathcal{A}, \quad \tau_T^{(\text{em})} = 0. \quad (5.17)$$

Hence, the Einstein equations for the Einstein-Maxwell system can be obtained by replacing τ_a in Eq. (5.10) by

$$\tau_a = \tau_a^{(\text{em})} + \bar{\tau}_a, \quad (5.18)$$

where the second term represents the contribution from matter other than the electromagnetic field.

5.3.2. Master equations

Now, generic modes: $m_V \neq 0$, we introduce new master variables by

$$\Phi_{\pm} := a_{\pm}r^{-n/2} \left(\tilde{\Omega} - \frac{2\kappa^2 q}{m_V} \mathcal{A} \right) + b_{\pm}r^{n/2-1}\mathcal{A}, \quad (5.19)$$

with

$$(a_+, b_+) = \left(\frac{Qm_V}{(n^2-1)M + \Delta}, \frac{Q}{q} \right), \quad (5.20a)$$

$$(a_-, b_-) = \left(1, \frac{-2n(n-1)Q^2}{q[(n^2-1)M + \Delta]} \right), \quad (5.20b)$$

where Δ is a positive constant satisfying

$$\Delta^2 = (n^2-1)^2 M^2 + 2n(n-1)m_V Q^2. \quad (5.21)$$

Then, from Eqs. (5.10) and (5.16), we obtain the two decoupled wave equations as our master equations:

$$\square \Phi_{\pm} - \frac{V_{V\pm}}{f} \Phi_{\pm} = S_{V\pm}, \quad (5.22)$$

where

$$V_{V\pm} = \frac{f}{r^2} \left[k_v^2 + \frac{(n^2-2n+4)K}{4} - \frac{n(n-2)}{4}\lambda r^2 + \frac{n(5n-2)Q^2}{4r^{2n-2}} + \frac{\mu_{\pm}}{r^{n-1}} \right], \quad (5.23)$$

$$\mu_{\pm} = -\frac{n^2+2}{2}M \pm \Delta, \quad (5.24)$$

and

$$S_{V\pm} = -a_{\pm} \frac{2\kappa^2 r^{n/2} f}{m_V} \epsilon^{ab} D_a(r\bar{\tau}_b) - b_{\pm} r^{n/2-1} f J. \quad (5.25)$$

For $n=2, K=1$ and $\lambda=0$, the variables Φ_+ and Φ_- are proportional to the variables for the axial modes, $Z_1^{(-)}$ and $Z_2^{(-)}$ given in Ref. 26), and V_{V+} and V_{V-} coincide with the corresponding potentials, $V_1^{(-)}$ and $V_2^{(-)}$, respectively.

Here, note that in the limit $Q \rightarrow 0$, Φ_+ becomes proportional to \mathcal{A} and Φ_- to Ω . Hence, Φ_+ and Φ_- represent the electromagnetic mode and the gravitational mode, respectively. In particular, in the limit $Q \rightarrow 0$, the equation for Φ_- coincides with the master equation for a vector perturbation on a neutral black hole background derived in Paper I.

As for the exceptional modes, $m_V = 0$, from the definition, Eq. (5.11), of $\tau^{(1)}$ and Eq. (5.17), we can express $\tau^{(1)}$ as

$$\tau^{(1)} = -q\mathcal{A} + \bar{\tau}^{(1)}. \quad (5.26)$$

Hence, Eq. (5.13) can be rewritten as

$$F^{(1)} = \frac{2\kappa^2(q\mathcal{A} - \bar{\tau}^{(1)})}{r^{n+1}}. \quad (5.27)$$

Inserting this into Eq. (5.16), we obtain

$$\frac{1}{r^{n-2}}D_a(r^{n-2}D^a\mathcal{A}) - \frac{1}{r^2}\left(2(n-1)K + \frac{2n(n-1)Q^2}{r^{2n-2}}\right)\mathcal{A} = -J - \frac{2\kappa^2q}{r^{2n}}\bar{\tau}^{(1)}. \quad (5.28)$$

Therefore, only the electromagnetic perturbation is dynamical.

§6. Scalar-type perturbations

6.1. General background case

In terms of scalar harmonics \mathbb{S} , we expand the scalar-type perturbation variables as

$$h_{ab} = f_{ab}\mathbb{S}, \quad h_a = -\frac{r}{k}f_a\mathbb{S}, \quad h_L = 2r^2H_L\mathbb{S}, \quad h_T = 2r^2\frac{H_T}{k^2}\mathbb{S}, \quad (6.1a)$$

$$\delta T_{ab} = \tau_{ab}\mathbb{S}, \quad \delta T_a = -\frac{r}{k}(Pf_a + \tau_a)\mathbb{S}, \quad (6.1b)$$

$$\delta T_L = r^2(2H_LP + \delta P)\mathbb{S}, \quad \delta T_T = \frac{r^2}{k^2}(2H_TP + \tau_T)\mathbb{S}, \quad (6.1c)$$

and $X_a = X_a\mathbb{S}$. Here P and all expansion coefficients, e.g. δP , are tensor fields on the m -dimensional spacetime \mathcal{N}^m .

The basis of the scalar-type gauge-invariant variables introduced in the previous section are expanded as:

$$F^{(0)}_{ab} = F_{ab}\mathbb{S}, \quad F^{(0)} = 2r^2F\mathbb{S}, \quad (6.2a)$$

$$\Sigma^{(0)}_{ab} = \Sigma_{ab}\mathbb{S}, \quad \Sigma^{(0)}_{ai} = r\Sigma_a\mathbb{S}_i, \quad (6.2b)$$

$$\Sigma^{(0)} = r^2\Sigma\mathbb{S}, \quad \Pi^{(0)}_{ij} = r^2\tau_T\mathbb{S}_{ij}. \quad (6.2c)$$

The expansion coefficients here $F_{ab}, F, \Sigma_{ab}, \Sigma_a, \Sigma, \tau_T$ as well as X_a are precisely the same as those given in Ref. 24).

For the exceptional modes with $k^2 = n$ for $K = 1$, h_T and δT_T (equivalently H_T and τ_T) are not defined, because a second-rank symmetric tensor cannot be

constructed from \mathbb{S} for these modes. In this case, we define $F, F_{ab}, \Sigma_{ab}, \Sigma_a$ and Σ_L by setting $X_L = 0$ in the above definitions. These quantities defined in this way are, however, gauge dependent. These exceptional modes are treated in Appendix C of Paper III.

6.2. Static black hole background

6.2.1. Maxwell equations

From now on we consider the static black hole background with $m = 2$. We often use ϵ_{ab} . We expand $\mathcal{E}^{(0)}$ and $\mathcal{E}_a^{(0)}$ defined in Eqs. (2.57), (2.58) as

$$\mathcal{E}\mathbb{S} = \mathcal{E}^{(0)}, \quad \mathcal{E}_a\mathbb{S} = -\frac{k}{r}\mathcal{E}_a^{(0)}. \quad (6.3a)$$

Then, the Maxwell equations (2.52) are written

$$\frac{1}{r^n}D_a(r^n\mathcal{E}) + \frac{k}{r}\mathcal{E}_a - \frac{E_0}{2}D_a(F_c^c - 2nF) = \epsilon_{ab}J^b, \quad (6.4a)$$

$$\epsilon^{ab}D_a(r^{n-1}\mathcal{E}_b) = -r^{n-1}J, \quad (6.4b)$$

with J defined by $J_i = rJ\mathbb{S}_i$. Note that from the first of Eqs. (2.51) we have the relation

$$\mathcal{E} = -\frac{1}{k}D_c(r\mathcal{E}^c). \quad (6.5)$$

Note also that Eqs. (6.4a) and (6.4b) give the current conservation law

$$D_c(r^n J^c) = -kr^{n-1}J. \quad (6.6)$$

We find that the gauge invariant variables, \mathcal{E}_a and \mathcal{E} , can be expressed in terms of the single master variable \mathcal{A} as

$$\mathcal{E}_a = \frac{k}{r^{n-1}}\left(D_a\mathcal{A} + \tilde{J}_a\right), \quad r^n\mathcal{E} = -k^2\mathcal{A} + \frac{q}{2}(F_c^c - 2nF), \quad (6.7)$$

where \tilde{J}_a has been defined by $J^a = k^2r^{-n}\epsilon^{ab}\tilde{J}_b$. By inserting these expressions into Eq. (6.5), we obtain the wave equation for \mathcal{A} :

$$r^{n-2}D_a\left(\frac{D^a\mathcal{A}}{r^{n-2}}\right) - \frac{k^2}{r^2}\mathcal{A} = -r^{n-2}D_a\left(\frac{\tilde{J}^a}{r^{n-2}}\right) - \frac{q}{2r^2}(F_c^c - 2nF). \quad (6.8)$$

The contribution of the electromagnetic field to the perturbation of the energy-momentum tensor, Σ_{ab} , Σ_a and Σ_L , are

$$\Sigma_{ab}^{(\text{em})} = \left(\frac{qk^2}{r^{2n}}\mathcal{A} + \frac{nq^2}{r^{2n}}F\right)g_{ab} - \frac{q^2}{2r^{2n}}F_{ab}, \quad (6.9a)$$

$$\Sigma_a^{(\text{em})} = -\frac{qk}{r^{2n-1}}\left(D_a\mathcal{A} + \tilde{J}_a\right), \quad (6.9b)$$

$$\Sigma^{(\text{em})} = -\frac{qk^2}{r^{2n}}\mathcal{A} - \frac{nq^2}{r^{2n}}F. \quad (6.9c)$$

6.2.2. Master equations

For generic modes of scalar perturbations, the Einstein equations consist of four sets of equations of the forms

$$E_{ab} = \kappa^2 \Sigma_{ab}, \quad E_a = \kappa^2 \Sigma_a, \quad E_L = \kappa^2 \Sigma, \quad E_T = \kappa^2 \tau_T. \quad (6.10)$$

For the definitions of E_{ab}, E_a, E_L and E_T , see eqs. (63)–(66) in Ref. 24). Introducing the perturbation of the energy-momentum tensor as

$$S_{ab} = r^{n-2} \kappa^2 (\Sigma_{ab} - \Sigma_{ab}^{(\text{em})}), \quad S_a = \frac{r^{n-1} \kappa^2}{k} (\Sigma_a - \Sigma_a^{(\text{em})}), \quad S_L = r^{n-2} \kappa^2 (\Sigma_L - \Sigma_L^{(\text{em})}), \quad (6.11)$$

and

$$S_T = \frac{2r^n}{k^2} \kappa^2 \tau_T. \quad (6.12)$$

the conservation law for the energy-momentum tensor is given by the two equations

$$\frac{1}{r^2} D_a (r^2 S^a) - S_L + \frac{(n-1)(k^2 - nK)}{2nr^2} S_T = 0, \quad (6.13a)$$

$$\frac{1}{r^2} D_b (r^2 S_a^b) + \frac{k^2}{r^2} S_a - n \frac{D_a r}{r} S_L = k^2 \frac{\kappa^2 q}{r^{n+2}} \tilde{J}_a. \quad (6.13b)$$

Now we introduce X, Y and Z defined by

$$X = r^{n-2} (F_t^t - 2F), \quad Y = r^{n-2} (F_r^r - 2F), \quad Z = r^{n-2} F_t^r, \quad (6.14)$$

as in Paper I. After the Fourier transformation with respect to the Killing time coordinate, t , of the black hole background, Eq. (2.14), the perturbation equations above can be reduced to a system consisting of three first order linear differential equations for X, Y, Z and a single linear algebraic constraint on them, with inhomogeneous terms given by $\mathcal{A}, J_a, S_{ab}, S_a, S_T$. As performed in Paper I, this constrained system can be further simplified to a single second-order ODE for a scalar field Φ given by

$$\Phi = - \frac{X + Y + S_T - nZ/i\omega r}{r^{n/2-2} H}, \quad (6.15)$$

with a source term, where

$$H = m + \frac{n(n+1)M}{r^{n-1}} - \frac{n^2 Q^2}{r^{2n-2}}, \quad (6.16)$$

$$m = k^2 - nK. \quad (6.17)$$

Thus obtained ODE for Φ and the Maxwell equation for \mathcal{A} , given by Eq. (6.8), form a coupled second-order ODEs with source terms. Then, by introducing new master variables, Φ_{\pm} , given as a linear combination of Φ and \mathcal{A} below, we obtain the following two decoupled master equations for scalar-type perturbations:

$$f \frac{d}{dr} \left(f \frac{d}{dr} \Phi_{\pm} \right) + (\omega^2 - V_{S\pm}) \Phi_{\pm} = S_{S\pm}, \quad (6.18)$$

where

$$\Phi_{\pm} := a_{\pm}\Phi + b_{\pm}\mathcal{A}, \quad (6.19)$$

with

$$(a_+, b_+) = \left(\frac{m}{n}Q + \frac{(n+1)(M+\mu)}{2r^{n-1}}Q, \frac{(n+1)(M+\mu)Q}{qr^{n/2-1}} \right), \quad (6.20a)$$

$$(a_-, b_-) = \left((n+1)(M+\mu) - \frac{2nQ^2}{r^{n-1}}, -\frac{4nQ^2}{qr^{n/2-1}} \right), \quad (6.20b)$$

and where μ is a positive constant satisfying

$$\mu^2 = M^2 + \frac{4mQ^2}{(n+1)^2}. \quad (6.21)$$

If we define the parameter δ by

$$\mu = (1 + 2m\delta)M, \quad (6.22)$$

the effective potentials $V_{S\pm}$ are expressed as

$$V_{S\pm} = \frac{fU_{\pm}}{64r^2H_{\pm}^2}, \quad (6.23)$$

with

$$H_+ = 1 - \frac{n(n+1)}{2}\delta x, \quad H_- = m + \frac{n(n+1)}{2}(1 + m\delta)x, \quad (6.24a)$$

$$x = \frac{2M}{r^{n-1}}, \quad y = \lambda r^2, \quad z = \frac{Q^2}{r^{2n-2}}, \quad (6.24b)$$

and

$$\begin{aligned} U_+ = & [-4n^3(n+2)(n+1)^2\delta^2x^2 - 48n^2(n+1)(n-2)\delta x \\ & - 16(n-2)(n-4)]y - \delta^3n^3(3n-2)(n+1)^4(1+m\delta)x^4 \\ & + 4\delta^2n^2(n+1)^2\{(n+1)(3n-2)m\delta + 4n^2 + n - 2\}x^3 \\ & + 4\delta(n+1)\{(n-2)(n-4)(n+1)(m+n^2K)\delta - 7n^3 + 7n^2 - 14n + 8\}x^2 \\ & + \{16(n+1)(-4m + 3n^2(n-2)K)\delta - 16(3n-2)(n-2)\}x \\ & + 64m + 16n(n+2)K, \end{aligned} \quad (6.25a)$$

$$\begin{aligned} U_- = & [-4n^3(n+2)(n+1)^2(1+m\delta)^2x^2 + 48n^2(n+1)(n-2)m(1+m\delta)x \\ & - 16(n-2)(n-4)m^2]y - n^3(3n-2)(n+1)^4\delta(1+m\delta)^3x^4 \\ & - 4n^2(n+1)^2(1+m\delta)^2\{(n+1)(3n-2)m\delta - n^2\}x^3 \\ & + 4(n+1)(1+m\delta)\{m(n-2)(n-4)(n+1)(m+n^2K)\delta \\ & + 4n(2n^2 - 3n + 4)m + n^2(n-2)(n-4)(n+1)K\}x^2 \\ & - 16m\{(n+1)m(-4m + 3n^2(n-2)K)\delta \\ & + 3n(n-4)m + 3n^2(n+1)(n-2)K\}x \\ & + 64m^3 + 16n(n+2)m^2K. \end{aligned} \quad (6.25b)$$

The source terms $S_{S\pm}$ are given by

$$S_{S\pm} = a_{\pm}\bar{S}_{\Phi} + b_{\pm}S_{\mathcal{A}}, \quad (6.26)$$

where $\bar{S}_{\Phi} = S_{\Phi}|_{\mathcal{A}=0}$ with

$$\begin{aligned} S_{\Phi} = \frac{f}{r^{n/2}H} & \left[\kappa^2 E_0 \left(\frac{P_{S1}}{H} \left(\mathcal{A} - \frac{\tilde{J}_t}{i\omega} \right) + 2nr f \tilde{J}_r + 2k^2 \frac{\tilde{J}_t}{i\omega} + 2nf \frac{r \partial_r \tilde{J}_t}{i\omega} \right) \right. \\ & - HS_T - \frac{P_{S2}}{H} \frac{S_t}{i\omega} - 2nf \frac{r \partial_r S_t}{i\omega} - 2nr f S_r \\ & \left. + \frac{P_{S3}}{H} \frac{r S_t^r}{i\omega} + 2r^2 \frac{\partial_r S_t^r}{i\omega} + 2r^2 S_r^r \right], \end{aligned} \quad (6.27)$$

and

$$S_A = - \left(\frac{2n^2(n-1)^2 z f^2}{r^2 H} + \omega^2 \right) \frac{\tilde{J}_t}{i\omega} - r^{n-2} f \partial_r \left(\frac{f \tilde{J}_r}{r^{n-2}} \right) + \frac{2(n-1)E_0}{i\omega H} f (nf S_t - r S_t^r). \quad (6.28)$$

Here P_{S1} , P_{S2} and P_{S3} are polynomials of x, y and z , whose explicit expressions are

$$\begin{aligned} P_{S1} = & [-4n^4 z + 2n^2(n+1)x - 4n(n-2)m] y \\ & + \{2n^2(n-1)x + 4n(n-2)m + 4n^3(n-2)K\} z \\ & - n^2(n^2-1)x^2 + \{-4n(n-2)m + 2n^2(n+1)K\} x \\ & + 4m^2 + 4n^2 m K, \end{aligned} \quad (6.29)$$

$$\begin{aligned} P_{S2} = & [6n^4 z - n^2(n+1)(n+2)x + 2n(n-4)m] y \\ & - 2n^4 z^2 + \{n^2(3n^2 - n + 2)x - 4n(n-2)m - 6n^3(n-1)K\} z \\ & - n^2(n+1)x^2 + \{n(3n-7)m + n^2(n^2-1)K\} x \\ & - 2m^2 - 2n(n-1)m K, \end{aligned} \quad (6.30)$$

$$P_{S3} = -2n^2(3n-2)z + n^2(n+1)x - 2(n-2)m. \quad (6.31)$$

Note that the master variable Φ_- coincides with that for the neutral and source-free case in Paper I for $Q = 0$ and $S_T = 0$. Also note that the following relations hold:

$$Q^2 = (n+1)^2 M^2 \delta(1+m\delta), \quad (6.32)$$

$$H = H_+ H_- . \quad (6.33)$$

From these relations, we find that $Q = 0$ corresponds to $\delta = 0$, and in this limit, Φ_- coincides with Φ , and its equation coincides with the mater equation for the master variable Φ derived in Paper I. Hence, Φ_- and Φ_+ represent the gravitational mode and the electromagnetic mode, respectively. For $n = 2$, $K = 1$ and $\lambda = 0$, these variables Φ_+ and Φ_- are proportional to the variables for the polar modes, $Z_1^{(+)}$ and $Z_2^{(+)}$, appearing in Ref. 26), and V_{S+} and V_{S-} coincide with the corresponding potentials, $V_1^{(+)}$ and $V_2^{(+)}$, respectively.

For the exceptional modes, the last of Eq. (6.10) is not obtained from the Einstein equations. However, this equation with $\tau_T = 0$ can be imposed as a gauge condition, as shown in Paper I. Under this gauge condition, all equations derived in this subsection hold without change. However, the variables still contain some residual gauge freedom. See Appendix D of Paper III, for how to eliminate the residual gauge freedom to extract physical degrees of freedom.

§7. Lovelock Black Holes

When Einstein derived the field equations for gravity, he adopted the three requirements as the guiding principle in addition to the principle of general relativity. The first is the metric ansatz that gravitational field is completely determined by the spacetime metric. The second is that the energy-momentum tensor sources gravity, hence it is balanced by a second-rank symmetric tensor constructed from the metric. The last is the requirement that the field equations contain the second derivatives of the metric at most and are quasi-linear, i.e., the coefficients of the second derivatives contain only the metric and its first derivatives. These requirements determine the gravitational field equation uniquely.

In 4-dimensions, we can obtain a similar result even if we loosen the third condition and require only that the field equations contain second derivatives of the metric at most. To be precise, gravity theories satisfying these weaker requirements are the Einstein gravity and the $f(R)$ gravity, the latter of which is mathematically equivalent to the Einstein gravity coupled with a scalar field with a non-trivial potential.

When we extend general relativity to higher dimensions, however, the difference between the two versions of the third requirement becomes important. In fact, we require the stronger version, we obtain the same field equations for gravity in higher dimensions. In contrast, we require the weaker version, we obtain a larger class of theories that contains the higher-dimensional general relativity as a special case. Such extensions to higher dimensions were first studied systematically by D. Lovelock in 1971.²⁷⁾ What he found was that the second-rank gravitational tensor E_{MN} balancing the energy-momentum tensor T_{MN} is a sum of polynomials in the curvature tensor and that the polynomial of each degree is unique up to a proportionality constant. The physical importance of such an extension was later recognised when B. Zwiebach²⁸⁾ pointed out that the special quadratic combinations of the curvature tensor named the Gauss-Bonnet term naturally arises when we add quadratic terms of the Ricci curvature to the quadratic term in the Riemann curvature tensor obtained as the α' correction to the field equations in the heterotic string theory to obtain a ghost-free theory. In this section, we briefly overview the Lovelock theory, its static black hole solution and perturbation theory recently developed by Takahashi and Soda.^{12), 29)}

For notational convenience, we introduce an orthonormal basis on \mathcal{M} , and throughout this section, we use upper case latin indices in the range A, B, \dots, J to label the 1-form θ^A , ($A = 0, \dots, D-1$), of the basis, while we use upper case latin indices in the range K, L, M, N, \dots to denote tensors on \mathcal{M} as in the rest of

this chapter.

7.1. Lovelock theory

In 1986, B. Zumino³⁰⁾ pointed out that the class of theories obtained by Lovelock has a natural mathematical meaning. That is, the Lovelock equations for gravity can be obtained from the action that is a linear combination of the terms each of which corresponds to the Euler form in some even dimensions:

$$S = \int \sum_{k=0}^{[(D-1)/2]} \alpha_k \mathcal{L}_k; \quad (7.1)$$

$$\begin{aligned} \mathcal{L}_k &:= \frac{1}{(D-2k)!} \epsilon_{A_1 \dots A_{2k} B_1 \dots B_{D-2k}} \mathcal{R}^{A_1 A_2} \wedge \dots \wedge \mathcal{R}^{A_{2k-1} A_{2k}} \wedge \theta^{B_1 \dots B_{D-2k}} \\ &= I_k \Omega_D, \end{aligned} \quad (7.2)$$

where \mathcal{R}^A_B is the curvature form with respect to the orthonormal 1-form basis θ^A ($A, B = 0, \dots, D-1$), $\theta^{A \dots B} = \theta^A \wedge \dots \wedge \theta^B$, Ω_D is the volume form, and

$$I_k = \frac{1}{2^k} \delta_{B_1 \dots B_{2k}}^{A_1 \dots A_{2k}} R_{A_1 A_2}^{B_1 B_2} \dots R_{A_{2k-1} A_{2k}}^{B_{2k-1} B_{2k}}. \quad (7.3)$$

The explicit expressions for small values of k are

$$I_0 = 1, \quad (7.4a)$$

$$I_1 = R, \quad (7.4b)$$

$$I_2 = R^2 - 4R_B^A R_A^B + R_{AB}^{CD} R_{CD}^{AB}, \quad (7.4c)$$

$$I_3 = R^3 - 3R(-4R_B^A R_A^B + R_{AB}^{CD} R_{CD}^{AB}) + 24R_B^A R_C^B R_A^C + 3R_{AB}^{CD} R_{CD}^{EF} R_{EF}^{AB}. \quad (7.4d)$$

Hence, if we require that the theory has the Einstein theory in the low energy limit, α_0 and α_1 are related to the cosmological constant Λ and the Newton constant $\kappa^2 = 8\pi G$ as

$$\alpha_0 = -\frac{\Lambda}{\kappa^2}, \quad \alpha_1 = \frac{1}{2\kappa^2}. \quad (7.5)$$

Further, if the Gauss-Bonnet term \mathcal{L}_2 comes from the $O(\alpha')$ correction in the heterotic string theory, $\alpha_2 > 0$.

From the Bianchi identity $\mathcal{D}\mathcal{R}_{AB} \equiv 0$, the variation of the Lagrangian density can be written

$$\begin{aligned} \delta \mathcal{L}_k &= d(*) + \frac{k}{(D-2k-1)!} \epsilon_{A_1 \dots A_{2k} B_1 \dots B_{D-2k}} \delta \omega^{A_1 A_2} \wedge \mathcal{R}^{A_3 A_4} \wedge \dots \wedge \theta^{B_1} \wedge \theta^{B_2 \dots B_{D-2k}} \\ &\quad + \frac{(-1)^{D-1}}{(D-2k-1)!} \epsilon_{A_1 \dots A_{2k} B_1 \dots B_{D-2k}} \mathcal{R}^{A_1 A_2} \wedge \dots \wedge \mathcal{R}^{A_{2k-1} A_{2k}} \wedge \theta^{B_1 \dots B_{D-2k-1}} \wedge \delta \theta^{B_{D-2k}} \\ &= d(*) + \left\{ k T^{(k)A}_{B_1 B_2} (\delta \omega^{B_1 B_2})_A + (-1)^{D-1} E^{(k)A}_B \delta \theta^B_M e_A^M \right\} \Omega_D, \end{aligned} \quad (7.6)$$

where ω^A_B is the connection form with respect to θ^A , e_A is the vector basis dual to θ^A , \mathcal{D} is the corresponding covariant exterior derivative, $\Theta^A = \mathcal{D}\theta^A$ is the torsion

2-form, and

$$T^{(k)A}_{B_1 B_2} = \delta_{B_1 B_2 D_1 \dots D_{2k-1}}^{AC_1 \dots C_{2k}} R^{D_1 D_2}_{C_1 C_2} \dots R^{D_{2k-3} D_{2k-2}}_{C_{2k-3} C_{2k-2}} T^{D_{2k-1}}_{C_{2k-1} C_{2k}} \quad (7.7a)$$

$$E^{(k)A}_B = -\frac{1}{2^k} \delta_{BB_1 \dots B_{2k}}^{AA_1 \dots A_{2k}} R_{A_1 A_2}{}^{B_1 B_2} \dots R_{A_{2k-1} A_{2k}}{}^{B_{2k-1} B_{2k}}. \quad (7.7b)$$

Hence, if we treat θ^A and ω^A_B as independent dynamical variables, the field equations are given by

$$\sum_k \alpha_k E^{(k)A}_B = 0, \quad (7.8a)$$

$$\sum_k k \alpha_k T^{(k)A}_{BC} = 0. \quad (7.8b)$$

If we require the connection to be Riemannian, the second equation becomes trivial due to the torsion free condition $T^A_{BC} = 0$. Examples of the explicit expressions for $E^{(k)A}_B$ and $T^{(k)A}_{BC}$ are

$$E^{(0)A}_B = -\delta^A_B, \quad (7.9a)$$

$$E^{(1)A}_B = 2R^A_B - R\delta^A_B, \quad (7.9b)$$

$$E^{(2)A}_B = -\delta^A_B I_2 + 4RR^A_B - 8R^{AC}{}_{BD} R^D_C \\ + 8R^A_C R^C_B + 4R^{AC_1 C_2 C_3} R_{BC_1 C_2 C_3}, \quad (7.9c)$$

$$T^{(1)A}_{BC} = T^A_{BC} + 2\delta^A_{[B} T^D_{C]D}, \quad (7.9d)$$

$$T^{(2)A}_{BC} = 8\delta^A_{[B} (-2R^D_{C]} T_D + R^{D_1 D_2}{}_{C] D_3} T^{D_3}_{D_1 D_2} + R T_{C]} - 2R^{D_1}_{D_2} T^{D_2}_{C] D_1}) \\ + 12(R^{AD}{}_{BC} T_D - 2R^A_{[B} T_{C]} - 2R^{AD_1}{}_{D_2 [B} T^{D_2}_{C] D_1} - 12R^A_D T^D_{BC}). \quad (7.9e)$$

7.2. Static black hole solution

7.2.1. Constant curvature spacetimes

In general relativity, a constant curvature spacetime is always a vacuum solution and the curvature is uniquely determined by the value of the cosmological constant. This feature is not shared by the Lovelock theory. In fact, the Lovelock theory does not allow a vacuum constant curvature solution for some range of the coupling constants $\{\alpha_k\}$, and have multiple constant curvature solutions with different curvatures for other range of the coupling constants.

To see this, let us insert the constant Riemann curvature

$$R_{MNLK} = \lambda(g_{ML}g_{NK} - g_{MK}g_{NL}) \quad (7.10)$$

into the field equation (7.7b). Then, we obtain

$$P(\lambda) = 0, \quad (7.11)$$

where

$$P(X) := \sum_{k=0}^{[(D-1)/2]} \alpha_k \frac{X^k}{(D-2k-1)!}. \quad (7.12)$$

For general relativity for which $\alpha_k = 0$ for $k \geq 2$, this equation has a unique solution. In contrast, when $D > 4$ and $\alpha_k \neq 0$ ($k \geq 2$), the equation can have no solution or multiple solutions depending on the functional shape of $P(X)$.

7.2.2. Black hole solution

Now, let us look for spherically symmetric black hole solutions. Because the Birkhoff-type theorem holds for the Lovelock theory except for the case in which $P(\lambda) = 0$ has a root with multiplicity higher than one,^{31),32)} we only consider static spacetimes whose metric can be put into the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\sigma_n^2, \quad (7.13)$$

where $d\sigma_n^2$ represents the metric of a constant curvature space with sectional curvature K . For a spherically symmetric solution, $K = 1$. However, because the argument in this section holds for any value of K , we consider this slightly general spacetime.

Then, the non-vanishing components of the curvature tensor are given up to symmetry by

$$R^{01}_{01} = \frac{h}{2} \left(-\frac{f''}{f} + \frac{(f')^2}{2f^2} \right) - \frac{h'f'}{4f}, \quad (7.14a)$$

$$R_{0i0j} = \frac{hf'}{2rf} g_{ij}, \quad R_{1i1j} = -\frac{h'}{2r} g_{ij}, \quad (7.14b)$$

$$R_{ijkl} = X(g_{ik}g_{jl} - g_{il}g_{jk}), \quad (7.14c)$$

where $X(r) := (K - h(r))/r^2$. Inserting these into Eq. (7.7b), we find that the field equations reduce to

$$(r^{n+1}P(X(r)))' = 0, \quad P^{(1)}(X(r))(f(r)/h(r))' = 0. \quad (7.15)$$

The first of these is integrated to yield

$$P(X(r)) = \frac{C}{r^{n+1}}, \quad (7.16)$$

where C is an integration constant. This determines the function $h(r)$ implicitly. In particular, for $C = 0$, we have $h(r) = K - \lambda r^2$ for each solution to $P(\lambda) = 0$. If $P^{(1)}(\lambda) \neq 0$, after an appropriate scaling of t , $f(r) = h(r)$ follows from the second of the above field equations:

$$ds^2 = -(K - \lambda r^2)dt^2 + \frac{dr^2}{K - \lambda r^2} + r^2 d\sigma_n^2. \quad (7.17)$$

This represents a constant curvature spacetime with sectional curvature λ irrespective of the value of K , as is well known.

This implies that for $C \neq 0$, we have in general multiple solutions corresponding to multiple solutions to $P(\lambda) = 0$. Each solution approaches a constant curvature

spacetime with sectional curvature λ at large r asymptotically. For these solutions, we can always put $f(r) = h(r)$ and the metric can be written³³⁾

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma_n^2, \quad f(r) = K - X(r)r^2. \quad (7.18)$$

In general, the constant C is proportional to the total mass M of the system and positive if $M > 0$. We can easily show that $X(r)$ changes monotonically with r from infinity to some value of r where the metric becomes singular. This singularity may or may not be hidden by a horizon depending on the functional shape of $P(X)$. In the former case, we obtain a regular black hole solution.

7.3. Perturbation equations for the static solution

The linear perturbation of the Lovelock tensor, Eq. (7.7b), in general reads

$$\delta E_N^M = - \sum_{k=1}^{[(D-1)/2]} \frac{k\alpha_k}{2^k} \delta_{NN_1 \dots N_{2k}}^{MM_1 \dots M_{2k}} R_{M_1 M_2}^{N_1 N_2} \dots R_{M_{2k-3} M_{2k-2}}^{N_{2k-3} N_{2k-2}} \delta R_{M_{2k-1} M_{2k}}^{N_{2k-1} N_{2k}}. \quad (7.19)$$

Inserting Eq. (7.14) with $f(r) = h(r) = K - X(r)r^2$ into this yields¹²⁾

$$r^{n-1} \delta E_t^t = -\frac{rT'}{n-1} \delta R_{ij}{}^{ij} - 2T \delta R_{ir}{}^{ir}, \quad (7.20a)$$

$$r^{n-1} \delta E_t^r = -2T \delta R_{it}{}^{ir}, \quad (7.20b)$$

$$r^{n-1} \delta E_r^r = -\frac{rT'}{n-1} \delta R_{ij}{}^{ij} - 2T \delta R_{it}{}^{it}, \quad (7.20c)$$

$$r^{n-1} \delta E_a^i = \frac{2rT'}{n-1} \delta R_{aj}{}^{ij} + 2T \delta R_{ab}{}^{ib}, \quad (7.20d)$$

$$r^{n-1} \delta E_j^i = \frac{2rT'}{n-1} \delta R_{aj}{}^{ai} + \frac{2r^2 T''}{(n-1)(n-2)} \delta R_{jk}{}^{ik} - \delta_j^i \left[2T \delta R_{tr}{}^{tr} + \frac{2rT'}{n-1} \delta R_{ak}{}^{ak} + \frac{r^2 T''}{(n-1)(n-2)} \delta R_{kl}{}^{kl} \right], \quad (7.20e)$$

where

$$T(r) := r^{n-1} P^{(1)}(X(r)). \quad (7.21)$$

Thus, we can obtain perturbation equations for the metric in the Lovelock theory simply by calculating the perturbation of the curvature tensor.

7.3.1. Tensor perturbations

For tensor perturbations, the metric perturbation can be expanded in terms of the harmonic tensor \mathbb{T}_{ij} as Eq. (4.1). The non-vanishing components of the curvature tensor for this type of perturbations read

$$\delta R_{ai}{}^{aj} = - \left(\square H_T + \frac{2}{r} Dr \cdot DH_T \right) \mathbb{T}_i^j, \quad (7.22a)$$

$$\delta R_{ik}{}^{jk} = \left[-(n-2) \frac{f'}{r} H_T' + \frac{2K + k_t^2}{r^2} H_T \right] \mathbb{T}_i^j. \quad (7.22b)$$

Hence, from the above expression for δE_i^j , we obtain the following wave equation for H_T :

$$\frac{1}{f}\ddot{H}_T - fH_T'' - \left(f\frac{T''}{T'} + \frac{2f}{r} + f'\right)H_T' + \frac{2K + k_t^2}{(n-2)r}\frac{T''}{T'}H_T = 0. \quad (7.23)$$

If we introduce the mode function $\Psi(r)$ by

$$H_T(t, r) = \frac{\Psi(r)}{r\sqrt{T'(r)}}e^{-i\omega t}, \quad (7.24)$$

this wave equation can be put into the standard form

$$-\frac{d^2\Psi}{dr_*^2} + V_t\Psi = \omega^2\Psi, \quad (7.25)$$

with the effective potential

$$V_t(r) = \frac{(2K + k_t^2)f}{(n-2)r}\frac{T''}{T'} + \frac{1}{r\sqrt{T'}}\frac{d^2(r\sqrt{T'})}{dr_*^2}, \quad (7.26)$$

where $dr_* = dr/f(r)$ as in the Einstein black hole case.

7.3.2. Vector perturbations

For vector perturbations, the perturbation of components of the curvature tensor that are relevant to the field equations can be expressed in terms of the basic gauge-invariant quantities F_a as

$$\delta R_{aj}{}^{ai} = -\frac{k}{r^2}D^a(rF_a)\mathbb{V}_j^i, \quad (7.27a)$$

$$\delta R_{jk}{}^{ik} = -\frac{(n-2)k}{r}\frac{D^a r}{r}F_a\mathbb{V}_j^i, \quad (7.27b)$$

$$\delta R_{aj}{}^{ij} = \frac{n-1}{2r^2}\left[-D^b F_{ba}^{(1)} - \frac{k^2 - (n-1)K}{(n-1)r}F_a + \frac{2(K-f) + rf'}{r}f_a\right]\mathbb{V}^i, \quad (7.27c)$$

$$\delta R_{ab}{}^{ib} = \left[-\frac{1}{2r^3}D^b(r^2F_{ba}^{(1)}) + \frac{rf'' - f'}{2r^2}f_a\right]\mathbb{V}^i. \quad (7.27d)$$

Inserting these into Eq. (7.20d), we obtain the following equations for the gauge-invariant variables:

$$\frac{1}{r^2}D^b\left(r^2TF_{ba}^{(1)}\right) + T'\frac{k^2 - (n-1)K}{(n-1)r}F_a = 0, \quad (7.28a)$$

$$\frac{1}{r}D^a(rT'F_a) = 0. \quad (7.28b)$$

The gauge-dependent residuals in the expressions for the curvature tensor cancel exactly owing to the identity

$$P^{(1)}X'' + P^{(2)}(X')^2 + \frac{n+2}{r}P^{(1)}X' = 0 \quad (7.29)$$

obtained from the background equation $(r^{n+1}P(X))' = 0$. We can easily confirm that for general relativity for which $T = r^{n-1}/(2\kappa^2)$, these reduce to Eqs. (5.2) and (5.3) with no source terms.

A master equation for vector perturbations in the Lovelock theory can be derived in the same way as that in general relativity. First, the second perturbation equation implies the existence of a potential Ω in which F_a can be expressed as

$$rT'F_a = \epsilon_{ab}D^b\Omega. \quad (7.30)$$

Inserting this into the first perturbation equation, we easily find that it is equivalent to

$$rTD_a\left(\frac{1}{r^2T'}D^a\Omega\right) - \frac{k^2 - (n-1)K}{(n-1)r^2}\Omega = 0. \quad (7.31)$$

7.3.3. Scalar perturbations

For scalar perturbations, we have

$$\begin{aligned} \delta R^{ai}{}_{bi} &= \frac{k^2}{2r^2}F_b^a + \frac{nf}{r}D_{[b}F_{r]}^a + \frac{n}{2r}D^aF_b^r + \frac{nf'}{2r}F_b^a - nD_bD^aF \\ &\quad - \frac{n}{r}(D^a_rD_b + D_brD^a)F + \frac{n}{2}X^cD_c\left(\frac{f'}{r}\right)\delta_b^a, \end{aligned} \quad (7.32a)$$

$$\begin{aligned} \delta R^{ai}{}_{aj} &= -\frac{k^2}{2r^2}F_a^i\mathbb{S}_j^i + \left\{\frac{D_ar}{r}D_bF^{ab} - \frac{1}{2r}Dr \cdot DF_a^a \right. \\ &\quad \left. + \left(\frac{f'}{2r} + \frac{k^2}{2nr^2}\right)F_a^a - \frac{1}{r^2}D^a(r^2D_aF) + X^aD_a\left(\frac{f'}{r}\right)\right\}\delta_j^i \mathbb{S}^i \end{aligned} \quad (7.32b)$$

$$\begin{aligned} \delta R^{ik}{}_{jk} &= (n-1)\left[\frac{D^a_rD^b_r}{r^2}F_{ab} - \frac{2}{r}Dr^aD_aF + \frac{2(k^2 - nK)}{nr^2}F \right. \\ &\quad \left. - D^b\left(\frac{K-f}{r^2}\right)X_b\right]\delta_j^i\mathbb{S}^i - (n-2)\frac{k^2}{r^2}F\mathbb{S}_j^i, \end{aligned} \quad (7.32c)$$

$$\delta R^{ib}{}_{ab} = \left[-\frac{k}{r}D_{[b}\left(\frac{1}{r}F_{a]}^b\right) + \frac{1}{2kr}(f' - rf'')D_aH_T\right]\mathbb{S}^i, \quad (7.32d)$$

$$\delta R^{ij}{}_{aj} = (n-1)k\left[-\frac{D_br}{2r^3}F_a^b + \frac{1}{r^2}D_aF - \frac{2(K-f) + rf'}{2r^2k^2}D_aH_T\right]\mathbb{S}^i, \quad (7.32e)$$

$$\delta R^{ab}{}_{ab} = \left\{-\square F_a^a + D^aD^bF_{ab} + \frac{f''}{2}F_a^a + X^aD_af''\right\}\mathbb{S}. \quad (7.32f)$$

Inserting these into Eq. (7.20), we obtain

$$\begin{aligned} r^{n-1}\delta E_t^t &\equiv \left[-\frac{nfT}{r}(F_r^r)' - \left(\frac{nf'}{r}T + \frac{nT'}{r}f + \frac{k^2}{r^2}T\right)F_r^r + 2nfTF'' \right. \\ &\quad \left. + \left(\frac{4nf}{r}T + 2nfT' + nf'T\right)F' - 2T'\frac{k^2 - nK}{r}F\right]\mathbb{S} = 0, \end{aligned} \quad (7.33a)$$

$$r^{n-1}\delta E_t^r \equiv \left[-\frac{nfT}{r}\left[\dot{F}_r^r + \frac{k^2}{nr}F_t^r - 2\sqrt{f}\left(\frac{r}{\sqrt{f}}\dot{F}\right)'\right]\right]\mathbb{S} = 0, \quad (7.33b)$$

$$r^{n-1}\delta E_r^r \equiv \left[\frac{Tnf}{r} \left\{ (F_t^t)' - \frac{k^2}{nfr} F_t^t \right\} - \frac{2nft}{r} \dot{F}_r^t - \frac{nfT}{r} \left(\frac{f'}{f} + \frac{T'}{T} \right) F_r^r - \frac{2nT}{f} \ddot{F} + nfT \left(\frac{f'}{f} + 2\frac{T'}{T} \right) F' - 2T' \frac{k^2 - nK}{r} F \right] \mathbb{S} = 0, \quad (7.33c)$$

$$r^{n-1}\delta E_a^i \equiv \frac{k}{r} \left[-D_b \left(\frac{T}{r} F_a^b \right) + TD_a \left(\frac{1}{r} F_b^b \right) + 2T' D_a F \right] \mathbb{S}^i = 0, \quad (7.33d)$$

$$r^{n-1}\delta E_j^i \equiv -\frac{k^2}{n-1} \left(\frac{T'}{r} F_a^a + 2T'' F \right) \mathbb{S}_j^i - \left[-D^a (TD_a F_b^b) + \left(\frac{(f'T)'}{2} + \frac{k^2 T'}{nr} \right) F_a^a + D^a D^b (TF_{ab}) - \frac{2}{r} D^a (r^2 T' D_a F) + \frac{2(k^2 - nK)}{n} T'' F \right] \delta_j^i \mathbb{S} = 0. \quad (7.33e)$$

As was first shown by Takahashi and Soda,¹²⁾ we can reduce these equations to a single master equation in terms of the master variable Φ defined by

$$F_t^r = r(\dot{\Phi} + 2\dot{F}), \quad (7.34)$$

as

$$\ddot{\Phi} - \frac{fA^2}{r^2 T'} \left(\frac{r^2 f T'}{A^2} \Phi' \right)' + Q\Phi = 0, \quad (7.35)$$

where

$$A = 2k^2 - 2nf + nrf', \quad (7.36a)$$

$$Q = \frac{f}{nr^2 T'} \left[r(k^2 T + nrfT') \left(2\frac{(AT)'}{AT} - \frac{T''}{T'} \right) - n(r^2 fT')' \right]. \quad (7.36b)$$

The other gauge-invariant variables are expressed in terms of Φ as

$$F = -\frac{1}{A} \left\{ nrf\Phi' + \left(k^2 + nrf\frac{T'}{T} \right) \Phi \right\}, \quad (7.37a)$$

$$F_r^r = -\frac{k^2}{nf} \Phi + 2rF' - \frac{A}{nf} F, \quad (7.37b)$$

$$F_t^t = -F_r^r - \frac{2rT''}{T'} F. \quad (7.37c)$$

§8. Stability Analysis

8.1. Stability criterion and S -deformation

With the decoupled master equations in hand, we are ready to study the stability of generalized static black holes with charge and cosmological constant in Einstein-Maxwell theory, whose metric is given by Eqs. (2.15) and (2.14). We consider only the stability in the static region outside the black horizon. This region is represented as $r > r_H$ for $\lambda \leq 0$ and $r_H < r < r_c$ for $\lambda > 0$. Such a region exists only for restricted ranges of the parameters M, Q and λ . [See Appendix A of Paper III for details.]

We have seen before that for any perturbation type, the master equations for perturbation in the static region are reduced to an eigenvalue problem of the type

$$\omega^2 \Phi = A\Phi, \quad (8.1)$$

where A is the derivative operator

$$A = -\frac{d^2}{dr_*^2} + V(r); \quad dr_* = \frac{dr}{f}, \quad (8.2)$$

with $V(r)$ being equal to $V_T(r)$, $V_{V\pm}(r)$ or $V_{S\pm}(r)$. The operator A is self-adjoint (by imposing suitable boundary conditions if necessary) in the standard square integrable function space $L^2(r_*, dr_*)$. Therefore, if its spectrum is non-negative, there is no exponentially growing mode among physically acceptable (i.e., normalizable) modes, implying the stability of the black hole.

When $\lambda \geq 0$, the static region where A is defined is globally hyperbolic and the range of r_* is complete. In this case the operator A is essentially self-adjoint with the domain of smooth functions of compact support, denoted hereafter by $C_0^\infty(r_*)$, and has the unique self-adjoint extension called the Friedrichs extension A_F , which is given by taking the closure of $(A, C_0^\infty(r_*))$ and is known to have the same lower bound of the spectrum of A with domain $C_0^\infty(r_*)$.

However, when $\lambda < 0$, r_* has an upper bound, and whether A becomes essentially self-adjoint depends upon the asymptotic behavior of the potential $V(r)$ in A , and thus upon the type of perturbations as well as the spacetime dimension. When A is not essentially self-adjoint, there are infinitely many different choices of boundary conditions that make A self-adjoint, and the spectrum of a self-adjoint extension depends upon the associated boundary conditions at the upper-bound of r_* ($r \rightarrow \infty$). In particular, even if A with $C_0^\infty(r_*)$ is positive-definite, its extension can admit a negative spectrum, depending on the choice of boundary conditions. This could be the case for the vector-type of electromagnetic and gravitational perturbations in $n = 2$ and for the scalar-type of electromagnetic and gravitational perturbations in $n = 2, 3, 4$, as shown in a simple, massless case by inspecting the asymptotic behavior of A .¹⁹⁾ For these cases, we must specify a boundary condition for Φ at $r = \infty$, and in the following we simply adopt the setting zero Dirichlet condition, $\Phi \rightarrow 0$ as $r \rightarrow \infty$, which corresponds to the Friedrichs extension, A_F .

Now suppose $\Phi(r)$ is a smooth function of compact support contained in $r > r_H$ (or $r_H < r < r_c$ for $\lambda > 0$). Then, we can rewrite the expectation value of A , $(\Phi, A\Phi)$, as

$$(\Phi, A\Phi) = \int dr_* \left(\left| \frac{d\Phi}{dr_*} \right|^2 + V|\Phi|^2 \right). \quad (8.3)$$

Note that no boundary terms appear as $\Phi \in C_0^\infty(r_*)$. When the static region is globally hyperbolic, we have the Friedrichs extension A_F . Then, in order to show the stability, it is sufficient to show the positivity of the right-hand side of Eq. (8.3) for $\Phi \in C_0^\infty(r_*)$,⁴⁾ since A_F has the same lower bound of $(A, C_0^\infty(r_*))$. In particular, this condition obviously is satisfied if V is non-negative. For 4-dimensional Schwarzschild

black hole, this is indeed the case. However, in higher dimensions, V is in general not positive definite, and it is far from obvious to know whether A is positive. One powerful method to show the positivity of A beyond such a simple situation is the procedure called the *S-deformation* of V in Papers II and III, in which we deform the right-hand side of Eq. (8.3) by partial integration in terms of a function S as

$$(\Phi, A\Phi) = \int dr_* \left(|\tilde{D}\Phi|^2 + \tilde{V}|\Phi|^2 \right), \quad (8.4)$$

where

$$\tilde{D} = \frac{d}{dr_*} + S, \quad \tilde{V} = V + f \frac{dS}{dr} - S^2, \quad (8.5)$$

and $\Phi \in C_0^\infty(r_*)$. Our task is now to find a suitable function S which makes the effective potential \tilde{V} positive. In the following we quote the results of the stability analysis of Papers II and III.

One might worry that our boundary conditions $\Phi \rightarrow 0$ in a neighbourhood of the horizon $r \rightarrow r_H$ would be too strong. However, if the horizon is non-degenerate and accordingly admits a bifurcate surface and once the stability is shown under our boundary conditions, then we can conclude by applying the theorem of Kay and Wald³⁴⁾ that the black hole spacetime is stable under perturbations that are non-vanishing at the bifurcate surface. Below we shall examine the stability for each type of perturbations.

8.2. Tensor perturbation

The stability analysis of higher dimensional static vacuum black holes under tensor type perturbations has first been examined by Gibbons and Hartnoll³⁵⁾ with special interests in the case where \mathcal{K}^n is a generic Einstein manifold. In the following we review our analysis in Paper III, which generalize their results.³⁵⁾

Consider the potential, Eq. (4.5). As discussed in Paper II, with the choice

$$S = -\frac{nf}{2r}, \quad (8.6)$$

the S -deformation yields

$$\tilde{V}_T = \frac{f}{r^2} [\lambda_L - 2(n-1)K], \quad (8.7)$$

irrespective of the r -dependence of $f(r)$. Therefore \tilde{V}_T becomes positive if

$$\lambda_L \geq 2(n-1)K. \quad (8.8)$$

In particular, this immediately guarantees the stability of maximally symmetric black holes for $K = 1$ and $K = 0$, since λ_L is related to the eigenvalue k_T^2 of the positive operator $-\hat{D}^i \hat{D}_i$ as $\lambda_L = k_T^2 + 2nK$ when \mathcal{K}^n is maximally symmetric. As for the case $K = -1$, \tilde{V}_T could be negative even in the maximally symmetric case, if $0 < k_T^2 < 2$. This is however not possible, due to the bound, Eq. (3.34). Therefore

we conclude that the maximally symmetric black holes considered here are stable under tensor type perturbations.

Note that the condition, Eq. (8.8), is just a sufficient condition for stability, and it is not a necessary condition in general. In fact, for tensor type perturbation, we can obtain stronger stability conditions directly from the positivity of V_T if we restrict the range of parameters. For example, for $K = 1$ and $\lambda = 0$, it is easy to see that V_T is positive if

$$\lambda_L + 2 - 2n + \frac{n(n-1)\sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}} \geq 0 \quad (8.9)$$

for $M^2 \geq Q^2 > 8n(n-1)M^2/(3n-2)^2$, and

$$\lambda_L + \frac{n^2 - 10n + 8}{4} \geq 0 \quad (8.10)$$

for $Q^2 \leq 8n(n-1)M^2/(3n-2)^2$. Thus, if we do not restrict the range of Q^2 , we obtain the same sufficient condition for stability as Eq. (8.8), but for the restricted range $Q^2 \leq 8n(n-1)M^2/(3n-2)^2$, we obtain the stronger sufficient condition, Eq. (8.10), which coincides with the condition obtained in Paper II for the case $Q = 0$.

Similarly, for $K = -1$ and $\lambda < 0$, if we restrict the range of λ to

$$\lambda \geq - \left(\frac{(n+1)M}{nQ^2} \right)^{\frac{2}{n-1}} \left(1 + \frac{(n^2-1)M^2}{n^2Q^2} \right), \quad (8.11)$$

we obtain from $V_T > 0$ a sufficient condition for stability stronger than Eq. (8.8),

$$\lambda_L + 3n - 2 = k_T^2 + n - 2 \geq 0. \quad (8.12)$$

This condition is sufficient to guarantee the stability of a maximally symmetric black hole with $K = -1$ for $n \geq 2$. However, if we extend the range of λ to the whole allowed range, then Eq. (8.8) becomes the strongest condition that can be obtained only from $V_T > 0$.

8.3. Vector perturbation

For the choice of

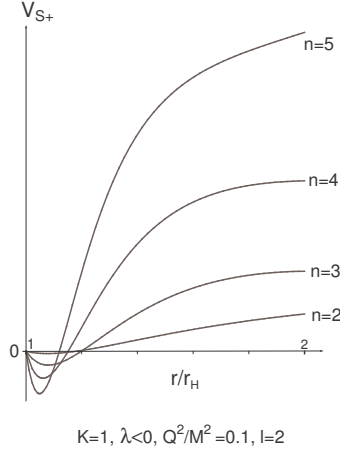
$$S = \frac{nf}{2r}, \quad (8.13)$$

$V = V_{V\pm}$ in Eq. (5.23) are deformed to

$$\tilde{V}_{V\pm} = \frac{f}{r^2} \left[m_V + \frac{(n^2-1)M \pm \Delta}{r^{n-1}} \right], \quad (8.14)$$

$$m_V = k_V^2 - (n-1)K. \quad (8.15)$$

It follows from $m_V \geq 0$ that \tilde{V}_{V+} is positive definite, and therefore static charged black holes are stable under the electromagnetic mode of vector type perturbation.


 Fig. 1. Examples of V_{S+} for $K = 1$ and $\lambda < 0$.

For the gravitational mode, we have

$$\tilde{V}_{V-} = \frac{f}{r^2} \frac{m_V h}{(n^2 - 1)M + \Delta}, \quad (8.16)$$

$$h := (n^2 - 1)M - \frac{2n(n-1)Q^2}{r^{n-1}} + \Delta. \quad (8.17)$$

Since h is a monotonically increasing function of r , \tilde{V}_{V-} is positive if and only if $h(r_H) \geq 0$. Therefore \tilde{V}_{V-} may become negative. In the case of $\lambda \geq 0$, the background spacetime contains a regular black hole only for $K = 1$, and the static region outside the black hole is given by $r_H < r < r_c (\leq +\infty)$. In this region, it turns out that $h > 0$ and therefore the black hole is stable. In the case of $\lambda < 0$, under the condition that the spacetime contains a regular black hole, we have the relation

$$h \geq \sqrt{(n^2 - 1)^2 M^2 + 2n(n-1)m_V Q^2} - \sqrt{(n^2 - 1)^2 M^2 - 4Kn(n-1)^2 Q^2}. \quad (8.18)$$

It follows that for $K = 0, 1$, $h > 0$. As for $K = -1$, the right-hand side of this inequality could become negative if $k_v^2 < n - 1$. This is however not possible as shown just below Eq. (3.22b). Therefore, $h > 0$ also for $K = -1$. We conclude that the black holes considered here are stable under vector type perturbations.

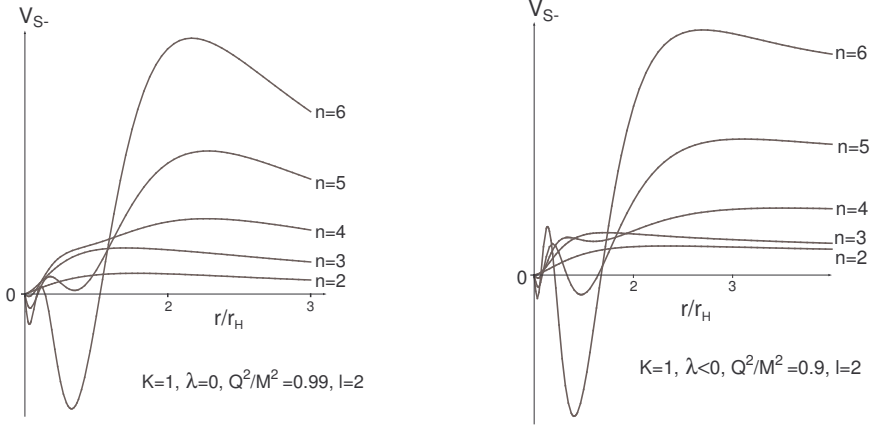
8.4. Scalar perturbation

By applying the S -deformation to V_{S+} with

$$S = \frac{f}{h_+} \frac{dh_+}{dr}, \quad h_+ = r^{n/2-1} H_+, \quad (8.19)$$

we obtain

$$\tilde{V}_{S+} = \frac{k^2 f}{2r^2 H_+} [(n-2)(n+1)\delta x + 2]. \quad (8.20)$$

Fig. 2. Examples of V_{S-} .

Since this is positive definite, the electromagnetic mode Φ_+ is always stable for any values of K , M , Q and λ , provided that the spacetime contains a regular black hole, although V_{S+} has a negative region near the horizon when $\lambda < 0$ and Q^2/M^2 is small (see Fig. 1).

Using a similar transformation, we can also prove the stability with respect to the gravitational mode Φ_- for some special cases. For example, the S -deformation of V_{S-} with

$$S = \frac{f}{h_-} \frac{dh_-}{dr}, \quad h_- = r^{n/2-1} H_- \quad (8.21)$$

leads to

$$\tilde{V}_{S-} = \frac{k^2 f}{2r^2 H_-} [2m - (n+1)(n-2)(1+m\delta)x]. \quad (8.22)$$

For $n = 2$, this is positive definite for $m > 0$. When $K = 1$, $\lambda \geq 0$ and $n = 3$ or when $\lambda \geq 0, Q = 0$ and the horizon is S^4 , we can show that $\tilde{V}_{S-} > 0$. Hence, in these special cases, the black hole is stable with respect to any type of perturbation.

However, for the other cases, \tilde{V}_{S-} is not positive definite for generic values of the parameters. The S -deformation used to prove the stability of neutral black holes in Paper II is not effective either. This is because V_{S-} is negative in the immediate vicinity of the horizon for the extremal and near extremal cases, as shown in Fig. 2, and the S -deformation cannot remove this negative region if S is a regular function at the horizon. Hence, the stability problem for these generic cases with $n \geq 3$ is left open.

Our results are summarized in Table I. As shown there, maximally symmetric black holes are stable with respect to tensor and vector perturbations over the entire parameter range. In contrast, for scalar type perturbation, we were not able to prove even the stability of asymptotically flat black holes with charge in generic dimensions, due to the existence of a negative region in the effective potential around the horizon in the extremal and near extremal cases.

In this regard, it should be noted that by extensive numerical studies, Konoplya

Table I. Stabilities of generalized static black holes. In this table, “ D ” represents the spacetime dimension, $n + 2$. The results for tensor perturbations apply only for maximally symmetric black holes, while those for vector and scalar perturbations are valid for black holes with generic Einstein horizons, except in the case with $K = 1, Q = 0, \lambda > 0$ and $D = 6$.

		Tensor		Vector		Scalar	
		$Q = 0$	$Q \neq 0$	$Q = 0$	$Q \neq 0$	$Q = 0$	$Q \neq 0$
$K = 1$	$\lambda = 0$	OK	OK	OK	OK	OK	$D = 4, 5$ OK $D \geq 6$?
	$\lambda > 0$	OK	OK	OK	OK	$D \leq 6$ OK $D \geq 7$?	$D = 4, 5$ OK $D \geq 6$?
	$\lambda < 0$	OK	OK	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?
$K = 0$	$\lambda < 0$	OK	OK	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?
$K = -1$	$\lambda < 0$	OK	OK	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?

and Zhidenko³⁶⁾ have shown that Schwarzschild-de Sitter black holes are stable in $D := 2 + n = 5, \dots, 11$. They also found that charged Schwarzschild-de Sitter black holes can be unstable if the electric charge and cosmological constant are large enough in $D \geq 7$.³⁷⁾ As for charged asymptotically anti-de Sitter (AdS) black holes, they have found no evidence for instability in $D = 5, \dots, 11$.³⁸⁾

8.5. Lovelock black holes

As we have seen in section 7, there are known exact solutions of static black holes in Lovelock theory, and the master equations for all types of perturbations of static vacuum black holes in general Lovelock theory have recently been derived by Takahashi and Soda.¹²⁾ Using the master equations, they have found that an asymptotically flat, static Lovelock black hole with small mass is unstable in arbitrary higher dimensions; it is unstable with respect to tensor type perturbations in even-dimensions^{29), 39)} and with respect to scalar type perturbations in odd-dimensions.⁴⁰⁾ The stability under vector type perturbations in all dimensions has also been shown⁴⁰⁾ by applying the S -deformation technique.

In fact, such an instability against tensor and scalar type perturbations, as well as the stability under vector type perturbations, have already been indicated by earlier work^{41)–44)} performed within the framework of second-order Lovelock theory, often called the Einstein-Gauss-Bonnet theory. For such a restricted class of Lovelock theory—though most generic in $D = 5, 6$, the master equations for metric perturbations have previously been derived by Dotti and Gleiser.^{41), 42)} A numerical analysis of the (in)stability of static black holes in Einstein-Gauss-Bonnet theory in dimensions $D = 5, \dots, 11$ has been performed in Ref. 45).

It is interesting to note that the instability found in small Lovelock black holes is typically stronger in short distance scales rather than long distance scale/low multipoles as one may expect. For example, for tensor type perturbations, there appears the eigenvalue of tensor harmonics on the horizon manifold as an overall factor in the effective potential term of Eq. (8-4), and it is therefore always possible

to make the potential term dominant by taking a sufficiently large eigenvalue of tensor harmonics. This implies that if the effective potential term can be negative, the right-hand side of Eq. (8.4), as a whole, can also be negative. This is shown to be the case when the mass is sufficiently small. A similar argument also applies to the case of scalar type perturbations at higher multipole moments.⁴⁰⁾

§9. Summary and Discussions

We have reviewed a gauge invariant formalism for gravitational and electromagnetic perturbations of static charged black holes with cosmological constant in higher dimensions and, as an application, the stability analysis using the master equations derived in the developed formalism. In section 2, we have started with considering a fairly generic class of background spacetimes defined by a warped product of an m -dimensional spacetime and an n -dimensional internal space, where the latter corresponds to the horizon cross-section manifold. We have explained how to decompose tensor fields in the background spacetime from the viewpoint of the internal space, and have seen that second-rank symmetric tensor fields or metric perturbations are decomposed into tensor, vector and scalar-types. We have then constructed manifestly gauge invariant variables for each type of perturbations. After that in section 3, we have introduced harmonic tensors on the internal space to expand the gauge invariant variables in terms of them. We have presented several theorems concerning basic properties of harmonic tensors, and also given explicit expressions of the harmonic tensors in terms of homogeneous coordinates. In subsequent sections 4-6, we have briefly described how to reduce the perturbed Einstein and Maxwell equations written in terms of the gauge-invariant variables to a set of decoupled master equations for a single scalar variable on the 2-dimensional background spacetime for the vector and scalar type perturbations. For the tensor type perturbations, there is no electromagnetic perturbation mode and the reduction can immediately be done to obtain the master equation on the generic m -dimensional spacetime. In the black hole background case, our master equations generalize to higher dimensions in a manifestly gauge-invariant manner the well-known Regge-Wheeler-Zerilli equations for Schwarzschild black holes, and Moncrief's equations^{46),47)} for Reissner-Nordström black holes, as well as the master equations given by Cardoso and Lemos⁴⁸⁾ that include cosmological constant in 4-dimensions. We have seen that by taking Fourier decomposition with respect to the time coordinate, each of the master equations is expressed in the form of a one-dimensional self-adjoint ODE. Therefore, as guaranteed by spectral theory of self-adjoint operators, the master equations can govern all possible perturbations that are normalizable with respect to the standard inner product. This is in particular important when we address the stability problem of a given solution, as the stability proof should be a statement concerning all physically acceptable (normalizable) perturbations. This is in contrast to the rotating black hole case, for which we have Teukolsky's equations in 4-dimensions and a similar set of master equations⁸⁾⁻¹⁰⁾ for some special cases of higher dimensional Myers-Perry black holes, but those master equations do not reduce to a form of the self-adjoint eigenvalue problem, except for some special modes [c.f., Ref. 25)]. In section 7,

we have also briefly reviewed a similar type of master equations for gravitational perturbations of static black holes in generic Lovelock theory, derived recently.¹²⁾

As an immediate and one of the most important applications of the master equations, we have examined the stability of higher dimensional static black holes with charge and cosmological constant in section 8. The task is to show the positivity of the self-adjoint operator appeared as the spatial derivative part of the master equations. For higher dimensional black holes, the potential term of the relevant self-adjoint operator is in general not positive definite, in contrast to the 4-dimensional case, and the stability is therefore not taken for granted in higher dimensions. To deal with such situations, we have developed the S -deformation technique, in which our task is to find some suitable function, S , that makes the effective potential deformed by S positive definite. Having applied this technique we have shown that a large variety of static black holes in higher dimensional general relativity are stable under gravitational as well as electromagnetic perturbations as summarized in Table I. This technique has also applied to the (in)stability analysis in general Lovelock theory, as has just been discussed in section 8.5 above. A similar type of stability analysis using the S -deformation, restricted to tensor-type perturbations, has also been performed for static black holes in higher derivative stringy gravity.⁴⁹⁾ Thus, the S -deformation has turned out to be a powerful tool to address stability problem.

However, as also indicated in Table I, the stability analysis of higher dimensional black holes has not been completed yet even within the context of general relativity. For some cases, in particular, when electric charge and cosmological constant are involved, we have not yet been able to draw definite conclusions for the stability problem with respect to scalar type perturbations. This is in part because there does not seem to be a systematic method to find such a desirable function S that could apply to generic cases. In particular, for the charged black hole case, the potential for scalar-type perturbation admits a negative ditch in the immediate vicinity of the horizon, which appears to be difficult to remove by the S -deformation. In connection to this, it would be interesting to note that the existence of such a negative ditch in the potential may have a significant influence on the frequencies of the quasinormal modes and the graybody factor for the Hawking process, even if these black holes are found to be stable. Also, given a choice of S , the stability would still depend upon the range of the parameters characterising the black hole solution as well as upon the eigenvalues of the harmonic functions. Therefore our analysis using the S -deformation needs to go on a case-by-case basis. A numerical analysis³⁸⁾ has found no indication of instability for charged AdS black holes in $D = 5, \dots, 11$. Therefore, at least for charged AdS black holes it may still be possible to analytically prove its stability by using the S -deformation or other analytic methods.

As other applications than the stability issue, the master equations can be used in numerical studies of black hole quasinormal modes in higher dimensions.^{48), 50)–54)} It would also be interesting to consider stationary perturbations that could describe deformation of the event horizon, as considered in Ref. 37) for unstable charged de Sitter black holes. If one finds no stationary perturbation that is regular everywhere on and outside the event horizon, then it would support the uniqueness property^{55), 56)} of the given background black hole solution, as analysed in Paper II

for the vacuum black hole case, as well as in Ref. 57) for more general cases. This may be interesting in particular in asymptotically AdS black hole case to find deformed horizon solutions [see Ref. 58) for such an analysis in 4-dimensions]. Also, we note that depending on the type of perturbations and the dimensionality, asymptotically AdS black holes admit a large class of boundary conditions at conformal infinity, other than the Dirichlet conditions considered in section 8. As has been considered in the context of gauge/gravity correspondence and often examined within AdS gravity coupled to a scalar field,^{59), 60)} it would be interesting to clarify whether different choice of boundary conditions leads to different consequences for the stability problem.

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